Getting physics from 3d gravity: What does an observer in 3d gravity see?

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Workshop “Chern-Simons Gauge Theory: 20 years after”
Hausdorff Center for Mathematics, Bonn

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Motivation
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(2+1)-gravity
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Physics: toy model for (quantum) gravity in higher dimensions
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  • black hole physics, AdS-CFT correspondence, ...
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**Relation to Chern-Simons theory** [Witten]
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gauge theoretical viewpoint
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rigorously in concrete example

• clarify: “quantum gravity as quantisation of geometry“?

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Why? physically relevant (sensible causality behaviour)
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6. Outlook and conclusions
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- complete set of diffeomorphism invariant observables:
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central role in quantisation of theory
Phase space and geometry: spacetimes as quotients of Minkowski space

[Mess], [Benedetti], [Barbot], [Bonsante], [Guadagnini],...
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Phase space and geometry: spacetimes as quotients of Minkowski space \[ \text{[Mess], [Benedetti], [Barbot], [Bonsante], [Guadagnini],...} \]

1. Universal cover = regular domain \[ D = \bigcup_{T \in \mathbb{R}^+} D_T \subset M^3 \]
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3. **Spacetime: quotient of domain by group action**
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Example: conformally static spacetimes
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quotient spacetime $M = \bigcup_{T \in \mathbb{R}^+} T \cdot \Sigma_g$
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Evolving spacetimes via grafting [Mess], [Thurston]
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\[ \diamond \text{ evolving spacetime} \]
- holonomies acquire non-trivial translational component
- geometry of cct-surfaces changes with cosmological time
Evolution with the cosmological time

Static spacetime

Grafted spacetime

\[ T \]
Phase space and observables
Phase space and observables

phase space
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phase space [Mess]
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Phase space and observables

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Wilson loops
Phase space and observables

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Wilson loops
-conj. inv. functions of holonomies along closed curves in M
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**Wilson loops**

- conjug. inv. functions of holonomies along closed curves in \( M \)
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**mass** \( m : \lambda \mapsto m_\lambda = |n_\lambda| \)
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- spacetimes isometric iff holonomies related by conjugation with constant element \( h_0 \in P_3 \)
  \[ \{ h'(\lambda) \mid \lambda \in \pi_1(M) \} = h_0 \cdot \{ h(\lambda) \mid \lambda \in \pi_1(M) \} \cdot h_0^{-1} \]

phase space \( \text{Hom}_0(\pi_1(M), P_3)/P_3 \)

Dirac observables functions on \( \text{Hom}_0(\pi_1(M), P_3)/P_3 \)
  \( \Leftrightarrow \) functions of the holonomies \( h(\lambda), \lambda \in \pi_1(M) \) invariant under simultaneous conjugation with \( P_3 \)

Wilson loops
- conjug. inv. functions of holonomies along closed curves in \( M \)
- two fundamental Wilson loop observables for \( \lambda \in \pi_1(M) \)
  \[ h(\lambda) = (\exp(n^a_\lambda J_a), a_\lambda) \]

mass \( m : \lambda \mapsto m_\lambda = |n_\lambda| \)
spin \( s : \lambda \mapsto s_\lambda = \hat{n}_\lambda \cdot a_\lambda \)
2. Measurements via returning light rays
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problems / questions
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problems / questions
- physical interpretation of observables (Wilson loops)?
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idea: observer emits “test light rays” that return to him/her for certain directions
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  eigentime elapsed between emission and return of lightray
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similar to gravitational lensing
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✧ “topological lensing”
3. Measurements via returning lightrays
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worldline of observer in free fall
3. Measurements via returning lightrays

worldline of observer in free fall

$\pi_1(M)$-equiv. class of timelike, future oriented geodesics in $D$
3. Measurements via returning light rays

worldline of observer in free fall

$\pi_1(M)$-equiv. class of timelike, future oriented geodesics in $D$
3. Measurements via returning light rays

Worldline of observer in free fall

$\pi_1(M)$-equiv. class of timelike, future oriented geodesics in $D$

**Parametrisation**

\[ h(\lambda)g = (v_\lambda, a_\lambda)g, \quad \lambda \in \pi_1(M) \]

\[ g(t) = t \cdot x + x_0 \quad x^2 = -1, \quad x_0 \in D \]
3. Measurements via returning light rays

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**Rapidity**

$cosh \rho_\lambda = x \cdot v_\lambda x$
3. Measurements via returning lightrays

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parametrisation

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g(t) = t \cdot x + x_0 \quad x^2 = -1, \quad x_0 \in D
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3 parameters for relative initial position
3. Measurements via returning lightrays

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$h(\lambda)g = (v_\lambda, a_\lambda)g$, $\lambda \in \pi_1(M)$

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**rapidity**  
$cosh \rho_\lambda = x \cdot v_\lambda x$

3 parameters for relative initial position

$h(\lambda)g(0) - g(0) = \sigma_\lambda (v_\lambda x - x) + \tau_\lambda v_\lambda x + \nu_\lambda x \wedge v_\lambda x$
3. Measurements via returning light rays

Worldline of observer in free fall

$\pi_1(M)$-equiv. class of timelike, future oriented geodesics in $\mathcal{D}$

**Parametrisation**

$h(\lambda)g = (v_\lambda, a_\lambda)g, \ \lambda \in \pi_1(M)$

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returning lightray
3. Measurements via returning light rays

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**parametrisation**

\[
\begin{align*}
    h(\lambda)g &= (v_\lambda, a_\lambda)g, \quad \lambda \in \pi_1(M) \\
g(t) &= t \cdot x + x_0 \\
x^2 &= -1, \quad x_0 \in D
\end{align*}
\]

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\end{align*}
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**returning light ray**

light ray emitted by observer at time \( t \) that returns at time \( t + \Delta t \)
3. Measurements via returning light rays

worldline of observer in free fall
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**returning light ray**

light ray emitted by observer at time \( t \) that returns at time \( t + \Delta t \)
\( \Leftrightarrow \) lightlike geodesic in \( D \) from \( g \) to image \( h(\lambda)g, \ \lambda \in \pi_1(M) \)
3. Measurements via returning lightrays

- Worldline of observer in free fall
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- Returning lightray
  \[ \text{lightray emitted by observer at time } t \text{ that returns at time } t + \Delta t \]
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**returning light ray**

light ray emitted by observer at time $t$ that returns at time $t+\Delta t$

$\Leftrightarrow$ lightlike geodesic in $D$ from $g$ to image $h(\lambda)g$, $\lambda \in \pi_1(M)$

$\Leftrightarrow$ 1:1-correspondence with elements of $\pi_1(M)$
3. Measurements via returning lightrays

**worldline of observer in free fall**

\[ \pi_1(M) \text{-equiv. class of timelike, future oriented geodesics in } D \]

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\[ h(\lambda)g = (v_\lambda, a_\lambda)g, \; \lambda \in \pi_1(M) \]

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**returning lightray**

Lightray emitted by observer at time \( t \) that returns at time \( t + \Delta t \)

\[ \Rightarrow \text{ lightlike geodesic in } D \text{ from } g \text{ to image } h(\lambda)g, \; \lambda \in \pi_1(M) \]

\[ \Rightarrow 1:1\text{-correspondence with elements of } \pi_1(M) \]

**condition**

\[ (h(\lambda)g(t + \Delta t) - g(t))^2 = 0, \; \lambda \in \pi_1(M) \]
Return time

\[ \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma \lambda)(\cosh \rho \lambda - 1) - \tau \lambda + \sinh \rho \lambda \sqrt{(t + \sigma \lambda)^2 + \nu^2_\lambda} \]
return time

$$\Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}$$

direction of emission
return time

\[
\Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}
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\[ \Delta t(t, \mathbf{x}, \mathbf{x}_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \]

direction of emission

\[ \hat{\mathbf{p}}_\lambda(t) = \hat{\Pi}_{x\perp}(h(\lambda)g(t + \Delta t) - g(t)) = \cos \phi \hat{\Pi}_{x\perp}(v_\lambda \mathbf{x}) + \sin \phi \frac{\mathbf{x} \wedge v_\lambda \mathbf{x}}{|\mathbf{x} \wedge v_\lambda \mathbf{x}|} \]
return time

$$\Delta t(t, x, x_0, h(\lambda)) = (t + \sigma \lambda)(\cosh \rho \lambda - 1) - \tau \lambda + \sinh \rho \lambda \sqrt{(t + \sigma \lambda)^2 + \nu_{\lambda}^2}$$

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$$\hat{p}_\lambda(t) = \hat{\Pi}_{x \perp} (h(\lambda)g(t + \Delta t) - g(t)) = \cos \phi \hat{\Pi}_{x \perp} (v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|}$$

$$\phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_{\lambda}}{\sinh \rho \lambda \sqrt{(t + \sigma \lambda)^2 + \nu_{\lambda}^2} + (t + \sigma \lambda) \cosh \rho \lambda} \right)$$
return time

\[ \Delta t(t, \mathbf{x}, \mathbf{x}_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu^2_\lambda} \]

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angles between directions of emission

\[ \phi(t, \mathbf{x}, \mathbf{x}_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu^2_\lambda} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]
\[ \Delta t(t, \mathbf{x}, \mathbf{x}_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \]

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\[ \hat{p}_\lambda(t) = \hat{\Pi}_{\mathbf{x}^\perp}(h(\lambda)g(t + \Delta t) - g(t)) = \cos \phi \hat{\Pi}_{\mathbf{x}^\perp}(v_\lambda \mathbf{x}) + \sin \phi \left( \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \right) \]
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angles between directions of emission
\[ \cos \Phi_{\lambda_1, \lambda_2} = \hat{p}_{\lambda_1}(t) \cdot \hat{p}_{\lambda_1}(t) \]
\[
\Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}
\]

**Direction of emission**

\[
\hat{p}_\lambda(t) = \hat{\Pi}_{x_\perp} (h(\lambda)g(t + \Delta t) - g(t)) = \cos \phi \hat{\Pi}_{x_\perp} (v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|}
\]

\[
\phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2 + (t + \sigma_\lambda) \cosh \rho_\lambda}} \right)
\]

**Angles between directions of emission**

\[
\cos \Phi_{\lambda_1, \lambda_2} = \hat{p}_{\lambda_1}(t) \cdot \hat{p}_{\lambda_1}(t)
\]

**Frequency shift**

\(\Rightarrow\) via relativistic Doppler effect
return time

$$\Delta t(t, \mathbf{x}, \mathbf{x}_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}$$

direction of emission

$$\hat{p}_\lambda(t) = \hat{\Pi}_{\mathbf{x} \perp} (h(\lambda)g(t + \Delta t) - g(t)) = \cos \phi \hat{\Pi}_{\mathbf{x} \perp} (v_\lambda \mathbf{x}) + \sin \phi \frac{\mathbf{x} \wedge v_\lambda \mathbf{x}}{|\mathbf{x} \wedge v_\lambda \mathbf{x}|}$$

$$\phi(t, \mathbf{x}, \mathbf{x}_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right)$$

angles between directions of emission

$$\cos \Phi_{\lambda_1, \lambda_2} = \hat{p}_{\lambda_1}(t) \cdot \hat{p}_{\lambda_1}(t)$$

frequency shift $\Rightarrow$ via relativistic Doppler effect

$$\frac{f_r}{f_e} = \frac{\mathbf{x} \cdot (h(\lambda)g(t + \Delta t) - g(t))}{v_\lambda \mathbf{x} \cdot (h(\lambda)g(t + \Delta t) - g(t))}$$
return time

\[ \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \]

direction of emission

\[ \hat{p}_\lambda(t) = \hat{\Pi}_{x\perp} (h(\lambda)g(t + \Delta t) - g(t)) = \cos \phi \hat{\Pi}_{x\perp} (v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \]

\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]

angles between directions of emission

\[ \cos \Phi_{\lambda_1, \lambda_2} = \hat{p}_{\lambda_1}(t) \cdot \hat{p}_{\lambda_1}(t) \]

frequency shift via relativistic Doppler effect

\[ \frac{f_r}{f_e} = \frac{x \cdot (h(\lambda)g(t + \Delta t) - g(t))}{v_\lambda x \cdot (h(\lambda)g(t + \Delta t) - g(t))} \]

\[ \frac{f_r}{f_e(t, x, x_0, h(\lambda))} = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda (t + \sigma_\lambda)} \]
return time

\[ \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \]

direction of emission

\[ \hat{p}_\lambda(t) = \hat{\Pi}_{x\perp}(h(\lambda)g(t + \Delta t) - g(t)) = \cos \phi \hat{\Pi}_{x\perp}(v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \]

\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]

angles between directions of emission

\[ \cos \Phi_{\lambda_1, \lambda_2} = \hat{p}_{\lambda_1}(t) \cdot \hat{p}_{\lambda_1}(t) \]

frequency shift \: \xrightarrow{\text{via relativistic Doppler effect}} \: \text{redshift}

\[ f_r/f_e = \frac{x \cdot (h(\lambda)g(t + \Delta t) - g(t))}{v_\lambda x \cdot (h(\lambda)g(t + \Delta t) - g(t))} \]

\[ f_r/f_e(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda(t + \sigma_\lambda)} \]
**return time**

\[ \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \]

**direction of emission**

\[ \hat{p}_\lambda(t) = \hat{\Pi}_{x \perp} (h(\lambda)g(t + \Delta t) - g(t)) = \cos \phi \hat{\Pi}_{x \perp} (v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \]

\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]

**angles between directions of emission**

\[ \cos \Phi_{\lambda_1, \lambda_2} = \hat{p}_{\lambda_1}(t) \cdot \hat{p}_{\lambda_1}(t) \]

**frequency shift** \( \rightarrow \) via relativistic Doppler effect

\[ f_r / f_e = \frac{x \cdot (h(\lambda)g(t + \Delta t) - g(t))}{v_\lambda x \cdot (h(\lambda)g(t + \Delta t) - g(t))} \]

\[ f_r / f_e(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda (t + \sigma_\lambda)} \]

\( \rightarrow \) functions of: emission time \( \Delta t \), observer \( x, x_0 \), holonomies \( h(\lambda) \)
Measurements - physical interpretation
Measurements - physical interpretation

Return time
\[ \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda) (\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \]

Frequency shift
\[ f_r / f_e(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda (t + \sigma_\lambda)} \]

Direction of emission
\[ \hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{x\perp}(v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \]
\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda} \frac{1}{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]
Measurements - physical interpretation

**Return time**  \[ \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma \lambda)(\cosh \rho \lambda - 1) - \tau \lambda + \sinh \rho \lambda \sqrt{(t + \sigma \lambda)^2 + \nu^2} \]

**Frequency shift**  \[ f_r/f_e(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma \lambda)^2 + \nu^2}}{\cosh \rho \lambda \sqrt{(t + \sigma \lambda)^2 + \nu^2} + \sinh \rho \lambda (t + \sigma \lambda)} \]

**Direction of emission**  \[ \hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{x\perp}(v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \]

\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho \lambda \sqrt{(t + \sigma \lambda)^2 + \nu^2} + (t + \sigma \lambda) \cosh \rho \lambda} \right) \]

conformally static spacetimes and “big bang” observers
Measurements - physical interpretation

**Return time** \( \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \)

**Frequency shift** \( f_r/f_e(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda(t + \sigma_\lambda)} \)

**Direction of emission** \( \hat{p}(t, x, x_0, h(\lambda)) = \cos \phi \hat{\Pi}_{x_\perp}(v_\lambda x) + \sin \phi \, \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \)

\( \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \)

Conformally static spacetimes and “big bang” observers
Measurements - physical interpretation

**Return time**  \[ \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \]

**Frequency shift**  \[ f_r/f_e(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda (t + \sigma_\lambda)} \]

**Direction of emission**  \[ \hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{x^\perp}(v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \]

\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]

**Conformally static spacetimes and “big bang” observers**  
\[ h(\lambda) = (v_\lambda, 0) \quad \forall \lambda \in \pi_1(M) \quad \sigma_\lambda = \tau_\lambda = \nu_\lambda = 0 \quad \forall \lambda \in \pi_1(M) \]
Measurements - physical interpretation

**Return time**
\[
\Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}
\]

**Frequency shift**
\[
f_r/f_e(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda (t + \sigma_\lambda)}
\]

**Direction of emission**
\[
\hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{x^\perp}(v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|}
\]
\[
\phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right)
\]

**Conformally static spacetimes and “big bang” observers**
\[
h(\lambda) = (v_\lambda, 0) \quad \forall \lambda \in \pi_1(M) \quad \sigma_\lambda = \tau_\lambda = \nu_\lambda = 0 \quad \forall \lambda \in \pi_1(M)
\]

**Return time linear in t**
\[
\Delta t = t(\exp(\rho_\lambda) - 1)
\]
Measurements - physical interpretation

**return time** \( \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_{\lambda}^2} \)

**frequency shift** \( f_r/f_e(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_{\lambda}^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_{\lambda}^2} + \sinh \rho_\lambda (t + \sigma_\lambda)} \)

**direction of emission** \( \hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{x \perp} (v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \)

\( \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_{\lambda}^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \)

**conformally static spacetimes and “big bang” observers**

\( h(\lambda) = (v_\lambda, 0) \quad \forall \lambda \in \pi_1(M) \quad \sigma_\lambda = \tau_\lambda = \nu_\lambda = 0 \quad \forall \lambda \in \pi_1(M) \)

**return time** linear in \( t \)

\( \Delta t = t(\exp(\rho_\lambda) - 1) \)

**directions** constant

\( \hat{p}_\lambda = \hat{\Pi}_{x \perp} (v_\lambda x) \)
Measurements - physical interpretation

**Return Time**  
\[ \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \]

**Frequency Shift**  
\[ \frac{f_r}{f_e}(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda (t + \sigma_\lambda)} \]

**Direction of Emission**  
\[ \hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{x\perp}(v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \]
\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]

**Conformally Static Spacetimes and “Big Bang” Observers**  
\[ h(\lambda) = (v_\lambda, 0) \quad \forall \lambda \in \pi_1(M) \quad \sigma_\lambda = \tau_\lambda = \nu_\lambda = 0 \quad \forall \lambda \in \pi_1(M) \]

**Return Time** linear in \( t \)  
\[ \Delta t = t(\exp(\rho_\lambda) - 1) \]

**Directions** constant  
\[ \hat{p}_\lambda = \hat{\Pi}_{x\perp}(v_\lambda x) \]

**Frequency Shift** constant  
\[ \frac{f_r}{f_e} = \exp(-\rho_\lambda) \]
general spacetimes and observers
general spacetimes and observers

return time \[ \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \]

frequency shift \[ \frac{f_r}{f_e}(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda (t + \sigma_\lambda)} \]

direction of emission \[ \hat{p}_\lambda(t) = \cos \phi \, \hat{\Pi}_{x_\perp} (v_\lambda x) + \sin \phi \, \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \]

\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]
general spacetimes and observers

return time \[ \Delta t(t, \mathbf{x}, \mathbf{x}_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \]

frequency shift \[ f_{r}/f_{e}(t, \mathbf{x}, \mathbf{x}_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda(t + \sigma_\lambda)} \]

direction of emission \[ \hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{x \perp}(v_\lambda \mathbf{x}) + \sin \phi \frac{\mathbf{x} \wedge v_\lambda \mathbf{x}}{|\mathbf{x} \wedge v_\lambda \mathbf{x}|} \]

\[ \phi(t, \mathbf{x}, \mathbf{x}_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]

- values for static spacetimes approached in limit \( t \rightarrow \infty \)
general spacetimes and observers

**Return time**  
\[ \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma \lambda) (\cosh \rho \lambda - 1) - \tau \lambda + \sinh \rho \lambda \sqrt{(t + \sigma \lambda)^2 + \nu^2_{\lambda}} \]

**Frequency shift**  
\[ \frac{f_r}{f_e}(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma \lambda)^2 + \nu^2_{\lambda}}}{\cosh \rho \lambda \sqrt{(t + \sigma \lambda)^2 + \nu^2_{\lambda}} + \sinh \rho \lambda (t + \sigma \lambda)} \]

**Direction of emission**  
\[ \hat{p}_\lambda(t) = \cos \phi \, \hat{\Pi}_{x \perp} (v_{\lambda x}) + \sin \phi \, \frac{x \wedge v_{\lambda x}}{|x \wedge v_{\lambda x}|} \]
\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_{\lambda}}{\sinh \rho \lambda \sqrt{(t + \sigma \lambda)^2 + \nu^2_{\lambda}} + (t + \sigma \lambda) \cosh \rho \lambda} \right) \]

- values for static spacetimes approached in limit \( t \to \infty \)
- non-linearity / time dependence due to parameter \( \nu_{\lambda} \)
general spacetimes and observers

return time \[ \Delta t(t, \mathbf{x}, \mathbf{x}_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu^2_\lambda} \]

frequency shift \[ \frac{f_r}{f_e}(t, \mathbf{x}, \mathbf{x}_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu^2_\lambda}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu^2_\lambda} + \sinh \rho_\lambda(t + \sigma_\lambda)} \]

direction of emission \[ \hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{\mathbf{x} \perp} (v_\lambda \mathbf{x}) + \sin \phi \frac{\mathbf{x} \wedge v_\lambda \mathbf{x}}{|\mathbf{x} \wedge v_\lambda \mathbf{x}|} \]
\[ \phi(t, \mathbf{x}, \mathbf{x}_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu^2_\lambda} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]

- values for static spacetimes approached in limit \( t \to \infty \)
- non-linearity / time dependence due to parameter \( \nu_\lambda \)
- reflects properties of evolving spacetimes (via grafting)
general spacetimes and observers

return time \( \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \)

frequency shift \( f_r/f_e(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda (t + \sigma_\lambda)} \)

direction of emission \( \hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{x^\perp} (v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \)

\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]

- values for static spacetimes approached in limit \( t \to \infty \)
- non-linearity / time dependence due to parameter \( \nu_\lambda \)
- reflects properties of evolving spacetimes (via grafting)
general spacetimes and observers

**return time** \( \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \)

**frequency shift** \( f_r / f_e(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda (t + \sigma_\lambda)} \)

**direction of emission** \( \hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{x\perp} (v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \)

\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]

- values for static spacetimes approached in limit \( t \to \infty \)
- non-linearity / time dependence due to parameter \( \nu_\lambda \)
- reflects properties of evolving spacetimes (via grafting)

- geodesic not deflected
general spacetimes and observers

return time
\[ \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \]

frequency shift
\[ \frac{f_r}{f_e}(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda (t + \sigma_\lambda)} \]

direction of emission
\[ \hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{\perp}(v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \]
\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]

- values for static spacetimes approached in limit \( t \to \infty \)
- non-linearity / time dependence due to parameter \( \nu_\lambda \)
- reflects properties of evolving spacetimes (via grafting)

-\[ \nu_v = 0 \]
- geodesic not deflected
- length increases by constant
general spacetimes and observers

return time \[ \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu^2_\lambda} \]

frequency shift \[ f_r/f_e(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu^2_\lambda}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu^2_\lambda} + \sinh \rho_\lambda (t + \sigma_\lambda)} \]

direction of emission \[ \hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{x\perp} (v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \]

\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu^2_\lambda} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]

- values for static spacetimes approached in limit \( t \to \infty \)
- non-linearity / time dependence due to parameter \( \nu_\lambda \)
- reflects properties of evolving spacetimes (via grafting)

\[ \nu_\nu = 0 \]

- geodesic not deflected
- length increases by constant
- \( \Delta t \) linear in \( t \)
general spacetimes and observers

return time
\[ \Delta t(t, \mathbf{x}, \mathbf{x}_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \]

frequency shift
\[ \frac{f_r}{f_e}(t, \mathbf{x}, \mathbf{x}_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda (t + \sigma_\lambda)} \]

direction of emission
\[ \mathbf{p}_\lambda(t) = \cos \phi \hat{\Pi}_{\mathbf{x} \perp}(\nu_\lambda \mathbf{x}) + \sin \phi \frac{\mathbf{x} \wedge \nu_\lambda \mathbf{x}}{|\mathbf{x} \wedge \nu_\lambda \mathbf{x}|} \]
\[ \phi(t, \mathbf{x}, \mathbf{x}_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]

- values for static spacetimes approached in limit \( t \to \infty \)
- non-linearity / time dependence due to parameter \( \nu_\lambda \)
- reflects properties of evolving spacetimes (via grafting)

\[ \nu_v = 0 \]

- geodesic not deflected
- length increases by constant
- \( \Delta t \) linear in \( t \)
- frequency shift constant
general spacetimes and observers

return time \[ \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu^2_\lambda} \]

frequency shift \[ f_r/f_e(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu^2_\lambda}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu^2_\lambda} + \sinh \rho_\lambda (t + \sigma_\lambda)} \]

direction of emission \[ \hat{p}_\lambda(t) = \cos \phi \widehat{\Pi}_{x\perp} (\nu_\lambda x) + \sin \phi \frac{x \wedge \nu_\lambda x}{|x \wedge \nu_\lambda x|} \]
\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu^2_\lambda} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]

- values for static spacetimes approached in limit \( t \to \infty \)
- non-linearity / time dependence due to parameter \( \nu_\lambda \)
- reflects properties of evolving spacetimes (via grafting)

\[ \nu_v = 0 \]
\[ \nu_v \neq 0 \]

- geodesic not deflected
- length increases by constant
- \( \Delta t \) linear in \( t \)
- frequency shift constant
general spacetimes and observers

**Return time**  \( \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma \lambda)(\cosh \rho \lambda - 1) - \tau \lambda + \sinh \rho \lambda \sqrt{(t + \sigma \lambda)^2 + \nu^2} \)

**Frequency shift**  \( f_r/f_e(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma \lambda)^2 + \nu^2}}{\cosh \rho \lambda \sqrt{(t + \sigma \lambda)^2 + \nu^2} + \sinh \rho \lambda (t + \sigma \lambda)} \)

**Direction of emission**  \( \hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{x\perp}(\nu \lambda x) + \sin \phi \frac{x \wedge \nu \lambda x}{|x \wedge \nu \lambda x|} \)

\[
\phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu \lambda}{\sinh \rho \lambda \sqrt{(t + \sigma \lambda)^2 + \nu^2} + (t + \sigma \lambda) \cosh \rho \lambda} \right)
\]

- values for static spacetimes approached in limit  \( t \to \infty \)
- non-linearity / time dependence due to parameter  \( \nu \lambda \)
- reflects properties of evolving spacetimes (via grafting)

- geodesic not deflected
- length increases by constant
- \( \Delta t \) linear in  \( t \)
- frequency shift constant
general spacetimes and observers

return time \[ \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \]

frequency shift \[ \frac{f_r}{f_e}(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda (t + \sigma_\lambda)} \]

direction of emission \[ \hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{x \perp}(v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \]
\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]

- values for static spacetimes approached in limit \( t \to \infty \)
- non-linearity / time dependence due to parameter \( \nu_\lambda \)
- reflects properties of evolving spacetimes (via grafting)

- \( \nu_v = 0 \)
- geodesic not deflected
- length increases by constant
- \( \Delta t \) linear in \( t \)
- frequency shift constant

- \( \nu_v \neq 0 \)
- geodesic deflected
- length increase varies with \( t \)
general spacetimes and observers

return time \( \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \left(\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}\right) \)

frequency shift \( f_r/f_e(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda (t + \sigma_\lambda)} \)

direction of emission \( \hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{x\perp}(v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \)

\( \phi(t, x, x_0, h(\lambda)) = \arctan\left(\frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda}\right) \)

- values for static spacetimes approached in limit \( t \to \infty \)
- non-linearity / time dependence due to parameter \( \nu_\lambda \)
- reflects properties of evolving spacetimes (via grafting)

\[ \begin{align*}
\nu_v &= 0 \\
\nu_v &\neq 0
\end{align*} \]

- geodesic not deflected
- length increases by constant
- \( \Delta t \) linear in \( t \)
- frequency shift constant
- geodesic deflected
- length increase varies with \( t \)
- \( \Delta t \) nonlinear in \( t \)
general spacetimes and observers

return time  \[ \Delta t(t, x, x_0, h(\lambda)) = (t + \sigma_\lambda)(\cosh \rho_\lambda - 1) - \tau_\lambda + \sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} \]

frequency shift  \[ f_r/f_e(t, x, x_0, h(\lambda)) = \frac{\sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2}}{\cosh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + \sinh \rho_\lambda (t + \sigma_\lambda)} \]

direction of emission  \[ \hat{p}_\lambda(t) = \cos \phi \hat{\Pi}_{x \perp} (v_\lambda x) + \sin \phi \frac{x \wedge v_\lambda x}{|x \wedge v_\lambda x|} \]
\[ \phi(t, x, x_0, h(\lambda)) = \arctan \left( \frac{\nu_\lambda}{\sinh \rho_\lambda \sqrt{(t + \sigma_\lambda)^2 + \nu_\lambda^2} + (t + \sigma_\lambda) \cosh \rho_\lambda} \right) \]

- values for static spacetimes approached in limit \( t \to \infty \)
- non-linearity / time dependence due to parameter \( \nu_\lambda \)
- reflects properties of evolving spacetimes (via grafting)

- geodesic not deflected
- length increases by constant
- \( \Delta t \) linear in \( t \)
- frequency shift constant

- geodesic deflected
- length increase varies with \( t \)
- \( \Delta t \) nonlinear in \( t \)
- frequency shift not constant
4. Measurements of observers vs Dirac observables
4. Measurements of observers vs Dirac observables

Measurements by observers vs phase space
4. Measurements of observers vs Dirac observables

Measurements by observers vs phase space

\[ \text{phase space} \quad \text{Hom}_0(\pi_1(M), P_3)/P_3 \]
4. Measurements of observers vs Dirac observables

Measurements by observers vs phase space

phase space $\text{Hom}_0(\pi_1(M), P_3)/P_3$

measurements by observer
4. Measurements of observers vs Dirac observables

Measurements by observers vs phase space

phase space \( \text{Hom}_0(\pi_1(M), P_3)/P_3 \)

measurements by observer \( \bowtie \) functions of: emission time \( t \)

observer \( x \in \mathbb{H}^2, x_0 \in D \) holonomies \( h(\lambda) \)
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\( \ni \) not functions on physical phase space, not Dirac observables
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Dirac observables via specification of observer
4. Measurements of observers vs Dirac observables

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Dirac observables via specification of observer

\( \Rightarrow \) specify observer with respect to spacetime geometry
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phase space $\text{Hom}_0(\pi_1(M), P_3)/P_3$

measurements by observer $\triangleright$ functions of: emission time $t$

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"comoving" observer for $\lambda \in \pi_1(M)$
4. Measurements of observers vs Dirac observables

Measurements by observers vs phase space

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**measurements by observer** $\Rightarrow$ functions of: emission time $t$

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$\Rightarrow$ not functions on physical phase space, not Dirac observables

Dirac observables via specification of observer

$\Rightarrow$ specify observer with respect to spacetime geometry

„comoving“ observer for $\lambda \in \pi_1(M)$

- velocity vector in plane stabilised by $v_\lambda \in SO(2, 1)$
  $x \cdot n_\lambda = 0 \quad v_\lambda = \exp(n^a_\lambda J_a)$
4. Measurements of observers vs Dirac observables

Measurements by observers vs phase space

phase space \( \text{Hom}_0(\pi_1(M), P_3)/P_3 \)

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measurements by observer $\rightsquigarrow$ functions of: emission time $t$

observer $x \in \mathbb{H}^2$, $x_0 \in D$ holonomies $h(\lambda)$

$\rightsquigarrow$ not functions on physical phase space, not Dirac observables

Dirac observables via specification of observer

$\rightsquigarrow$ specify observer with respect to spacetime geometry

"comoving" observer for $\lambda \in \pi_1(M)$

• velocity vector in plane stabilised by $v_\lambda \in SO(2, 1)$
  
  $x \cdot n_\lambda = 0 \quad v_\lambda = \exp(n_\lambda^a J_a)$

• relative initial position vector orthogonal to plane stabilised by $v_\lambda \in SO(2, 1)$
  
  $n_\lambda \wedge (h(\lambda)g(0) - g(0)) = 0$
Measurements and Wilson loops
Measurements and Wilson loops

measurements of comoving observers
Measurements and Wilson loops

measurements of comoving observers

- **return time** \[ \Delta t(t, h(\lambda)) = (\cosh m_\lambda - 1)t + \sqrt{\sinh^2 m_\lambda t^2 + s_\lambda^2} \]

- **direction of emission** \[ \hat{p}_\lambda(t) = \cos \phi(t) n_\lambda \wedge x + \sin \phi(t) n_\lambda \]
  \[ \phi(t, h(\lambda)) = \arctan \left( \frac{s}{\sinh m_\lambda \cosh m_\lambda t + \sinh m_\lambda \sqrt{\sinh^2 m_\lambda t^2 + s_\lambda^2}} \right) \]

- **redshift** \[ \frac{f_r}{f_e}(t, h(\lambda)) = \frac{\sqrt{\sinh^2 m_\lambda t^2 + s_\lambda^2}}{\cosh m_\lambda \sqrt{\sinh^2 m_\lambda t^2 + s_\lambda^2 + t \sinh^2 m_\lambda}} \]
Measurements and Wilson loops

measurements of comoving observers

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• redshift \( f_r/f_e(t, h(\lambda)) = \frac{\sqrt{\sinh^2 m_\lambda t^2 + s_\lambda^2}}{\cosh m_\lambda \sqrt{\sinh^2 m_\lambda t^2 + s_\lambda^2} + t \sinh^2 m_\lambda} \)

⇒ given by the two fundamental Wilson loop observables

\( h(\lambda) = (\exp(n_\lambda^a J_a), a_\lambda) \)

mass \( m : \lambda \mapsto m_\lambda = |n_\lambda| \)

spin \( s : \lambda \mapsto s_\lambda = \hat{n}_\lambda \cdot a_\lambda \)
Measurements and Wilson loops

measurements of comoving observers

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\]

• redshift

\[
\frac{f_r}{f_e}(t, h(\lambda)) = \frac{\sqrt{\sinh^2 m_\lambda t^2 + s^2_\lambda}}{\cosh m_\lambda \sqrt{\sinh^2 m_\lambda t^2 + s^2_\lambda + t \sinh^2 m_\lambda}}
\]

given by the two fundamental Wilson loop observables

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h(\lambda) = (\exp(n^a_\lambda J_a), a_\lambda)
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functions of emission time \( t \) and physical phase space
Measurements and Wilson loops

measurements of comoving observers

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\( \Rightarrow \) given by the two fundamental Wilson loop observables

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**mass** \( m : \lambda \mapsto m_\lambda = |\mathbf{n}_\lambda| \)

**spin** \( s : \lambda \mapsto s_\lambda = \hat{\mathbf{n}}_\lambda \cdot a_\lambda \)

\( \Rightarrow \) functions of emission time \( t \) and physical phase space

\( \Rightarrow \) Wilson loops characterise measurements of specific observers
Role of time
Role of time

time in general relativity
Role of time

**time in general relativity**

- Hamiltonian is constraint $\Rightarrow$ no time evolution on phase space
Role of time

time in general relativity

• Hamiltonian is constraint $\Rightarrow$ no time evolution on phase space
• physical (Dirac) observables time-independent
Role of time

time in general relativity

• Hamiltonian is constraint $\Rightarrow$ no time evolution on phase space
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• but: measurements by observers generally involve time
Role of time

time in general relativity

- Hamiltonian is constraint ⇨ no time evolution on phase space
- physical (Dirac) observables time-independent
- but: measurements by observers generally involve time measurements (return time, angles, red shift)
Role of time

time in general relativity

• Hamiltonian is constraint $\Rightarrow$ no time evolution on phase space
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• but: measurements by observers generally involve time

measurements (return time, angles, red shift)

• measurements depend on physical state and on eigentime of observer at emission of light ray
Role of time

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Role of time

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- measurements depend on physical state and on eigentime of observer at emission of lightray
- eigentime of observer **not** function on physical phase space but external parameter
- time measurements as functions on the physical phase space: time elapsed between two physical events

**example:** eigentime elapsed between two measurements of return time that yield values $\Delta t(v) = c_1, \Delta t(v) = c_2$
Role of time

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example: eigentime elapsed between two measurements of return time that yield values $\Delta t(v) = c_1, \Delta t(v) = c_2$

question: which time to consider in quantum theory?
5. Reconstructing spacetime geometry
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Qu: How to determine physical state from measurements?
5. Reconstructing spacetime geometry

Qu: How to determine physical state from measurements?

phase space \( \text{Hom}_0(\pi_1(M), P_3)/P_3 \)
5. Reconstructing spacetime geometry

Qu: How to determine physical state from measurements?

phase space \( \text{Hom}_0(\pi_1(M), P_3)/P_3 \)

\( \Rightarrow \) determining physical state = determining holonomies \( h(\lambda_i) \in P_3 \)

for a set of generators \( \{\lambda_i\}_{i=1,...,k} \) of \( \pi_1(M) \) (up to conjugation)
5. Reconstructing spacetime geometry

**Qu: How to determine physical state from measurements?**

Phase space \( \text{Hom}_0(\pi_1(M), P_3)/P_3 \)

- determining physical state = determining holonomies \( h(\lambda_i) \in P_3 \) for a set of generators \( \{\lambda_i\}_{i=1,\ldots,k} \) of \( \pi_1(M) \) (up to conjugation)

**Question:**
Can this be done via measurements by an observer
5. Reconstructing spacetime geometry

Qu: How to determine physical state from measurements?

phase space \( \text{Hom}_0(\pi_1(M), P_3)/P_3 \)

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Question:
Can this be done via measurements by an observer
- ignorant of his own state of motion?
5. Reconstructing spacetime geometry

**Qu: How to determine physical state from measurements?**

*phase space* \( \text{Hom}_0(\pi_1(M), P_3)/P_3 \)

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**Question:**
Can this be done via measurements by an observer
  - ignorant of his own state of motion?
  - in a finite amount of eigentime?
5. Reconstructing spacetime geometry

Qu: How to determine physical state from measurements?

Phase space: $\text{Hom}_0(\pi_1(M), P_3)/P_3$

Determining physical state = determining holonomies $h(\lambda_i) \in P_3$ for a set of generators $\{\lambda_i\}_{i=1,...,k}$ of $\pi_1(M)$ (up to conjugation).

Question:
Can this be done via measurements by an observer
- ignorant of his own state of motion?
- in a finite amount of eigentime?

Answer: yes
5. Reconstructing spacetime geometry

How to determine physical state from measurements?

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Can this be done via measurements by an observer

- ignorant of his own state of motion?
- in a finite amount of eigentime?  

Answer: yes

Example: conformally static spacetime
5. Reconstructing spacetime geometry

**Qu:** How to determine physical state from measurements?

**Phase space** \( \text{Hom}_0(\pi_1(M), P_3)/P_3 \)

\(\Rightarrow\) determining physical state = determining holonomies \( h(\lambda_i) \in P_3 \)
for a set of generators \( \{\lambda_i\}_{i=1,\ldots,k} \) of \( \pi_1(M) \) (up to conjugation)

**Question:**
Can this be done via measurements by an observer

- ignorant of his own state of motion?
- in a finite amount of eigentime? \[\text{Answer: yes}\]

**Example:** conformally static spacetime

- translational part of holonomies vanishes (up to conjugation)
5. Reconstructing spacetime geometry

Qu: How to determine physical state from measurements?

**phase space** $\text{Hom}_0(\pi_1(M), P_3)/P_3$

- determining physical state = determining holonomies $h(\lambda_i) \in P_3$
  
  for a set of generators $\{\lambda_i\}_{i=1,...,k}$ of $\pi_1(M)$ (up to conjugation)

**Question:**
Can this be done via measurements by an observer

- ignorant of his own state of motion?
- in a finite amount of eigentime?  
  
  **Answer:** yes

**Example: conformally static spacetime**

- translational part of holonomies vanishes (up to conjugation)
- for big bang observer: reconstructing holonomies
5. Reconstructing spacetime geometry

Qu: How to determine physical state from measurements?

Phase space $\text{Hom}_0(\pi_1(M), P_3)/P_3$

Determining physical state = determining holonomies $h(\lambda_i) \in P_3$
for a set of generators $\{\lambda_i\}_{i=1,\ldots,k}$ of $\pi_1(M)$ (up to conjugation)

Question:
Can this be done via measurements by an observer
  • ignorant of his own state of motion?
  • in a finite amount of eigentime?

Answer: yes

Example: conformally static spacetime
  • translational part of holonomies vanishes (up to conjugation)
  • for big bang observer: reconstructing holonomies
    = reconstructing a set of generators of $\Gamma$ from measurements
5. Reconstructing spacetime geometry

Qu: How to determine physical state from measurements?

**phase space** \( \text{Hom}_0(\pi_1(M), P_3)/P_3 \)

- determining physical state = determining holonomies \( h(\lambda_i) \in P_3 \)
  for a set of generators \( \{\lambda_i\}_{i=1,...,k} \) of \( \pi_1(M) \) (up to conjugation)

**Question:**
Can this be done via measurements by an observer

- ignorant of his own state of motion?
- in a finite amount of eigentime?

**Answer:** yes

**Example: conformally static spacetime**

- translational part of holonomies vanishes (up to conjugation)
- for big bang observer: reconstructing holonomies
  = reconstructing a set of generators of \( \Gamma \) from measurements
  = reconstructing Dirichlet region of \( \Gamma \) from measurements
Dirichlet region
Dirichlet region

\[ R_D(\Gamma, \mathbf{x}) \quad = \quad \{ \mathbf{y} \in \mathbb{H}^2 \mid d(\mathbf{y}, \mathbf{x}) \leq d(\mathbf{y}, v\mathbf{x}) \quad \forall v \in \Gamma \} \]

\[ = \bigcap_{v \in \Gamma} D_v(\mathbf{x}) \quad D_v(\mathbf{x}) = \{ \mathbf{y} \in \mathbb{H}^2 \mid d(\mathbf{y}, \mathbf{x}) \leq d(\mathbf{y}, v\mathbf{x}) \} \]
Dirichlet region

\[ R_D(\Gamma, x) = \{ y \in \mathbb{H}^2 \mid d(y, x) \leq d(y, vx) \ \forall v \in \Gamma \} \]

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- geodesic arc 2k-gon (k\geq2g)
Dirichlet region

\[ R_D(\Gamma, x) = \{ y \in \mathbb{H}^2 | d(y, x) \leq d(y, vx) \ \forall v \in \Gamma \} \]

\[ = \bigcap_{v \in \Gamma} D_v(x) \quad D_v(x) = \{ y \in \mathbb{H}^2 | d(y, x) \leq d(y, vx) \} \]

- geodesic arc 2k-gon (k ≥ 2g)
- sides identified pairwise by set of generators of \( \Gamma \)
Dirichlet region

\[ R_D(\Gamma, \mathbf{x}) = \{ \mathbf{y} \in \mathbb{H}^2 \mid d(\mathbf{y}, \mathbf{x}) \leq d(\mathbf{y}, v\mathbf{x}) \ \forall v \in \Gamma \} \]

\[ = \bigcap_{v \in \Gamma} D_v(\mathbf{x}) \quad D_v(\mathbf{x}) = \{ \mathbf{y} \in \mathbb{H}^2 \mid d(\mathbf{y}, \mathbf{x}) \leq d(\mathbf{y}, v\mathbf{x}) \} \]

- geodesic arc 2k-gon (k≥2g)
- sides identified pairwise by set of generators of \( \Gamma \)

Measurements
Dirichlet region

\[ R_D(\Gamma, x) = \{ y \in \mathbb{H}^2 | d(y, x) \leq d(y, vx) \ \forall v \in \Gamma \} = \bigcap_{v \in \Gamma} D_v(x) \]

- geodesic arc 2k-gon (k\geq2g)
- sides identified pairwise by set of generators of \( \Gamma \)

Measurements
- observer emits light rays in all directions at time t
Dirichlet region

\[ R_D(\Gamma, x) = \{ y \in \mathbb{H}^2 | d(y, x) \leq d(y, vx) \ \forall v \in \Gamma \} \]

\[ = \bigcap_{v \in \Gamma} D_v(x) \quad D_v(x) = \{ y \in \mathbb{H}^2 | d(y, x) \leq d(y, vx) \} \]

- geodesic arc 2k-gon (k\geq2g)
- sides identified pairwise by set of generators of \( \Gamma \)

Measurements

- observer emits light rays in all directions at time \( t \)
- light rays return to him one by one
**Dirichlet region**

\[ R_D(\Gamma, x) = \{ y \in \mathbb{H}^2 | d(y, x) \leq d(y, vx) \forall v \in \Gamma \} \]

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- geodesic arc 2k-gon (k \geq 2g)
- sides identified pairwise by set of generators of \( \Gamma \)

**Measurements**

- observer emits light rays in all directions at time \( t \)
- light rays return to him one by one
- observer measures for each returning light ray
Dirichlet region

\[ R_D(\Gamma, x) = \{ y \in \mathbb{H}^2 \mid d(y, x) \leq d(y, vx) \ \forall v \in \Gamma \} \]

\[ = \bigcap_{v \in \Gamma} D_v(x) \]

\[ D_v(x) = \{ y \in \mathbb{H}^2 \mid d(y, x) \leq d(y, vx) \} \]

- geodesic arc 2k-gon (k≥2g)
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**Measurements**

- observer emits light rays in all directions at time \( t \)
- light rays return to him one by one
- observer measures for each returning light ray

  • return time \( \Delta t = t \cdot (\exp(\rho_v) - 1) \) \( \Rightarrow \) distance \( \rho_v = d(x, vx) \)
Dirichlet region
\[ R_D(\Gamma, x) = \{ y \in \mathbb{H}^2 | d(y, x) \leq d(y, v x) \ \forall v \in \Gamma \} \]
\[ = \bigcap_{v \in \Gamma} D_v(x) \]
- geodesic arc 2k-gon (k \geq 2g)
- sides identified pairwise by set of generators of \( \Gamma \)

Measurements
- observer emits light rays in all directions at time \( t \)
- light rays return to him one by one
- observer measures for each returning lightray
  - return time \( \Delta t = t \cdot (\exp(\rho_v) - 1) \Rightarrow \text{distance} \ \rho_v = d(x, v x) \)
  - direction
Dirichlet region
\[ R_D(\Gamma, x) = \{ y \in \mathbb{H}^2 | d(y, x) \leq d(y, vx) \ \forall v \in \Gamma \} \]
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- geodesic arc 2k-gon (k \geq 2g)
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Measurements
- observer emits light rays in all directions at time \( t \)
- light rays return to him one by one
- observer measures for each returning lightray
  - return time \( \Delta t = t \cdot (\exp(\rho_v) - 1) \) \( \Rightarrow \) distance \( \rho_v = d(x, vx) \)
  - direction
  \( \Rightarrow \) for each returning lightray associated with \( v \in \Gamma \):
    location of image \( vx \in \mathbb{H}^2 \)
Reconstructing the Dirichlet region from measurements
Reconstructing the Dirichlet region from measurements

- observer assigns his velocity vector \( \mathbf{x} \) to origin on disc
Reconstructing the Dirichlet region from measurements

- observer assigns his velocity vector $\mathbf{x}$ to origin on disc
Reconstructing the Dirichlet region from measurements

- observer assigns his velocity vector $\mathbf{x}$ to origin on disc
- for each returning lightray:
Reconstructing the Dirichlet region from measurements

- observer assigns his velocity vector $\mathbf{x}$ to origin on disc
- for each returning light ray:
  • observer draws image $\mathbf{v}\mathbf{x}$
Reconstructing the Dirichlet region from measurements

- observer assigns his velocity vector $\mathbf{x}$ to origin on disc
- for each returning light ray:
  - observer draws image $\mathbf{v}x$
Reconstructing the Dirichlet region from measurements

- observer assigns his velocity vector $\mathbf{x}$ to origin on disc

- for each returning light ray:
  - observer draws image $v\mathbf{x}$
  - observer constructs perpendicular bisector
Reconstructing the Dirichlet region from measurements

- observer assigns his velocity vector $\mathbf{x}$ to origin on disc
- for each returning lightray:
  - observer draws image $\mathbf{v}\mathbf{x}$
  - observer constructs perpendicular bisector
Reconstructing the Dirichlet region from measurements

- observer assigns his velocity vector $\mathbf{x}$ to origin on disc
- for each returning lightray:
  - observer draws image $\mathbf{v}\mathbf{x}$
  - observer constructs perpendicular bisector
- after a finite number of returning lightrays: geodesic arc polygon
Reconstructing the Dirichlet region from measurements

- observer assigns his velocity vector $\mathbf{x}$ to origin on disc
- for each returning lightray:
  - observer draws image $\mathbf{v}\mathbf{x}$
  - observer constructs perpendicular bisector
- after a finite number of returning lightrays: geodesic arc polygon
Reconstructing the Dirichlet region from measurements

- observer assigns his velocity vector $\mathbf{x}$ to origin on disc

- for each returning lightray:
  
  - observer draws image $v\mathbf{x}$
  
  - observer constructs perpendicular bisector

- after a finite number of returning lightrays:
  geodesic arc polygon
Reconstructing the Dirichlet region from measurements

- observer assigns his velocity vector $\mathbf{x}$ to origin on disc
- for each returning lightray:
  - observer draws image $\mathbf{v}\mathbf{x}$
  - observer constructs perpendicular bisector
- after a finite number of returning lightrays:
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$\triangleright$ Dirichlet region and set of generators of $\Gamma$ in finite eigentime
$\triangleright$ full geometry of spacetime in finite eigentime
Reconstructing the Dirichlet region from measurements: general ignorant observer in general spacetime
Reconstructing the Dirichlet region from measurements: general ignorant observer in general spacetime

independence of observer’s reference frame
Reconstructing the Dirichlet region from measurements: general ignorant observer in general spacetime

**Independence of observer’s reference frame**

different choice of velocity vector $x \in \mathbb{H}^2$
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5. Outlook and Conclusions
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maximal globally hyperbolic genus $g \geq 2$ vacuum spacetimes in Lorentzian (2+1)-gravity, $\Lambda = 0$
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<tr>
<th>(2+1)-gravity isometry group</th>
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  - $\theta = 1$
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### (2+1)-gravity

#### Isometry group

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#### Gauge group

### Lie algebra

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Einstein Hilbert action

\[
S[g] = \int_M \sqrt{\det g} (R - 2\Lambda)
\]
(2+1)-gravity isometry group Chern-Simons theory gauge group

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**Chern-Simons action**

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\[
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(2+1)-gravity

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