Functional Integrals in Low-Dimensional Gauge Theories

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an example of the ‘unreasonable effectiveness’ of physics in mathematics.

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On the right is a mathematically meaningful quantity which can be studied independently of the left.

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- Perturbative
- Non-perturbative definition of the functional integral.

Functional Integrals in Gauge Theories

- Chern-Simons functional integral

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- Yang-Mills functional integral

$$\frac{1}{Z_g} \int_{\mathcal{A}} f(A) e^{-\frac{1}{2g^2} S_{\mathrm{YM}}(A)} \, \mathcal{D}A$$

Lebesgue Measure in Infinite Dimensions

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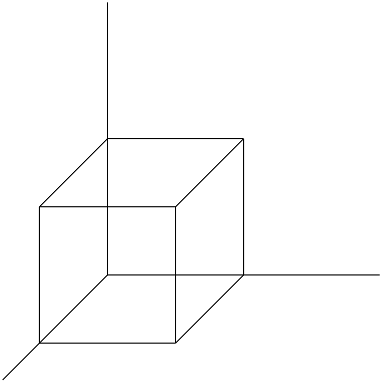
There is no useful form of this measure in infinite dimensions.



There are uncountably many translates of the unit cube in $\mathbb{R}^{\{1,2,3,...\}}$

A ‘flat’ Lebesgue measure on $\mathbb{R}^{\{1,2,3,\ldots\}}$ is not σ-finite, i.e. $\mathbb{R}^{\{1,2,3,\ldots\}}$ is the uncountable union of sets of finite positive measure.
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It is no fun doing useful analysis with non-σ-finite measures.
Fubini's theorem on interchanging integrals

$$\int \left[\int f \, dx \right] \, dy = \int \left[\int f \, dy \right] \, dx$$

is not guaranteed.

$$\text{Lebesgue measure}([0,a]^\infty) = \begin{cases} \infty & \text{if } a > 1; \\ 1 & \text{if } a = 1; \\ 0 & \text{if } a < 1; \end{cases}$$

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Do you really want to integrate with a measure like this?

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Such a metric could arise from a metric on the underlying bundle on which the connections are defined.

Constructing Integrals Rigorously

Let V be an infinite-dimensional vector space. One approach to a rigorously meaningful formulation of an integral of the form

$$\int_V f(x)\, e^{-\beta S(x)}\, Dx$$

is to think of it as a linear functional

$$\Phi : f \mapsto \Phi(f)$$

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$$\Phi : f \mapsto \Phi(f)$$

for f in a suitably large class of functions "on V".
Formal calculations specify what $\Phi(f)$ 'should' be for some good class of functions f.

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Φ might come from integration with respect to a genuine *measure*, or it might be a *distribution*.

Gaussian Measure

Let V be an infinite-dimensional real Hilbert space.
One would like to to work with integrals of the form

$$\Phi(f) = Z^{-1} \int_V f(y) e^{-\frac{|y|_V^2}{2}} \, Dy$$

where Z is a normalizing constant, ensuring that $\Phi(1)$ is 1.

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where Z is a normalizing constant, ensuring that $\Phi(1)$ is 1.
(In)formal computation shows that

$$\Phi(e^{i\langle \cdot, x \rangle}) = e^{-\|x\|_V^2/2}$$

Gaussian Measure

It is known that there is probability measure μ on a certain dual V' of V, such that

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$$\Phi(f) = \int f\, d\mu,$$

all rigorously meaningful.

If V is finite-dimensional then μ is the standard Gaussian measure

$$d\mu(x) = (2\pi)^{-\dim V/2} e^{-\|x\|^2/2}\, dx.$$

PS to Gaussian

If we are looking for a rigorous formulation for

$$\Phi : f \mapsto \int f(y)\, e^{-\beta S(y)}\, dy$$

then it is good news if Φ turns out to satisfy

$$\Phi(e^{\langle x, \cdot \rangle}) = e^{\text{quadratic in x}}$$

Distributions vs Measures

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This is the case for the Chern-Simons functional integral.

Chern-Simons for $U(1)$

Recall that the Chern-Simons action has the form

$$\mathrm{CS}(A) = \int_M \langle A \wedge dA \rangle + \text{cubic wedge term}$$

When the gauge group is abelian the cubic term drops out.

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(but careful: there is an i in the exponent still!).

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providing a rigorous formulation of the expression of certain toplogical invariants in terms of Chern-Simons integrals.

(Fresnel integral formulations of many Feynman path integrals are developed in the book by Albeverio, Høegh-Krohn, and Mazzucchi, Springer LNM 2008.)

Chern-Simons for $U(1)$

Interesting recent work of Guadagnini and Thuillier (2008) on abelian Chern-Simons theory needs to be examined by distributional methods.

Non-abelian Chern-Simons for $\mathbb{R}^3$

Consider now a connection for a possibly non-abelian gauge group G.
On $\mathbb{R}^3$ with coordinates (x_0, x_1, x_2), we can work in a gauge choice in which the connection is of the form

$$A = a_0 dx_0 + a_1 dx_1 + 0 dx_2$$

The component a_2 is set to 0 by gauge transforming the original A.

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The component a_2 is set to 0 by gauge transforming the original A.
The advantage is that again $\mathrm{CS}(A)$ is quadratic:

$$\mathrm{CS}(a_0, a_1) = \langle a_0, -\partial_2 a_1 \rangle,$$

where the inner-product on the right is $L^2(\mathbb{R}^3; \mathrm{Lie}(G))$.

Non-abelian Chern-Simons for $\mathbb{R}^3$

A formal calculation shows that

$$\int' e^{i\langle b_0,a_0\rangle + i\langle b_1,a_1\rangle} e^{i\mathrm{CS}(A)} da_0 da_1 = e^{-i\frac{1}{2}Q^{\mathrm{ax}}(b,b)},$$

where $\int'$ is the normalized formal integral, and

$$Q^{\mathrm{ax}}(b,b) = \langle (b_0,b_1), \begin{pmatrix} 0 & -\partial_2 \\ \partial_2 & 0 \end{pmatrix}^{-1} (b_0,b_1)\rangle$$

with $\partial_2^{-1} f$ chosen appropriately.

Non-abelian Chern-Simons for $\mathbb{R}^3$

There is, in fact, a rigorously meaningful distribution Φ_{CS}, i.e. a continuous linear functional on a space of functions on $\mathcal{A}$, such that

$$\Phi_{\mathrm{CS}}\left(e^{i\langle b_0,\cdot\rangle + i\langle b_1,\cdot\rangle}\right) = e^{-i\frac{1}{2}Q^{\mathrm{ax}}(b,b)}.$$

[Albeverio and S. CMP (1997)]

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Technical warning: more honestly, Φ_{CS} is defined on a space of functions on $\mathcal{A}'$, which is a completion of $\mathcal{A}$ gauge-fixed.
The big question is, of course, can we relate this to topological invariants?

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The answer is most likely, no.
It is too much to ask for.

Regularization and Wilson Loops

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$$\mathrm{Hol}(l_\epsilon; A)$$

(matrix groups here)

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(matrix groups here)
Secondly, deform Q^{ax} by a diffeomorphism ϕ_s of $\mathbb{R}^3$:

$$Q^{\mathrm{ax}}_{\phi_s}(b) = Q^{\mathrm{ax}}(b, (\phi_s)_* b)$$

Regularization and Wilson Loops

For a link L comprised of loops $l_1, ..., l_m$, work out

$$\Phi_{\mathrm{CS},\phi_s}(L, \epsilon)$$

where on the left the quadratic form $Q^{\mathrm{ax}}_{\phi_s}$ is used instead of Q^{ax}, and the loops l_j are smeared into tubes of thickness ϵ.

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A. Hahn (2002; CMP 2004) showed that

$$\lim_{\epsilon\downarrow 0, s\downarrow 0} \Phi_{\mathrm{CS},\phi_s}(L,\epsilon)$$

exists and expressed this limit in terms of link invariants.

Regularization and Wilson Loops

The diffeomorphisms ϕ_s involves some choices related to frames for links.

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No axial gauge choice here, of course.
Insead the idea of *torus gauge* fixing (Blau-Thompson) is used.

The two-dimensional Yang-Mills Integral

Yang-Mills functional integrals have the form

$$\langle f(A) \rangle_g = \frac{1}{Z_g} \int_{\mathcal{A}} f(A) e^{-\frac{1}{2g^2} S_{\mathrm{YM}}(A)} DA$$

where functions f of interest are generally products of traces of holonomies $h(c; A)$ of the connection A around loops c.

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The action S_{YM} is

$$S_{\mathrm{YM}}(A) = \int \|F^A\|^2\, d\sigma,$$

where $d\sigma$ is the volume measure on the base manifold M over which the gauge fields A are defined.

The two-dimensional Yang-Mills Integral

Summary

When the underlying manifold M is of dimension 2 there is a rigorously defined probability measure μ_g on a space $\mathcal{A}'$ of generalized connections such that, with $f(A)$ being a product of traces of holonomies around loops,

$$\langle f(A) \rangle_g = \int f(A)\, d\mu_g(A)$$

has exactly the values predicted in the physics literature (Witten, Migdal, Polyakov, Kazakov, Kostov, Bralic, et al.).

The two-dimensional Yang-Mills Integral

There are several approaches to the two-dimensional Yang-Mills functional integral.

Note: Here we discuss only pure Yang-Mills.

Fine's bundle calculation

Fine worked out Wilson loop expectation values by viewing $\mathcal{A}/\mathcal{G}_o$ as a bundle over, for instance, a space of paths in the gauge group G^{2h}, where h is the genus of the surface. (CMP 1991)

Stochastic construction of YM_2

The measure μ_g is constructed as a Gaussian measure in infinite dimensions conditioned to satisfy certain topological constraints imposed by the underlying base manifold (surface) and bundle topology.

[S. (Memoirs AMS 1997)]

With respect to this measure, parallel-transport is described by a *stochastic differential equation*, and its solution leads to concrete formulas for the Wilson loop expectation values.

earlier work of Driver (CMP 1989), Gross, King, S. (Ann. Phys. 1989) for plane

Stochastic construction of YM_2

The strategy very briefly:

$$\mathcal{A}/\mathcal{G}_o \simeq \mathcal{M} \subset L^2\big(D; \mathrm{Lie}(G)\big) \times G^{2h}$$

where $\mathcal{M}$ is an infinite-dimensional submanifold.
Then μ_g comes out as

$$\text{Gauss on } L^2 \times \text{ Haar on } G^{2h}$$

conditioned to satisfy topological and geometric constraints as expressed by means of solutions to s.d.e.

Probability theory of the Yang-Mills measure

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the large-deviations principle is governed by the YM action.

Large N

It has been known for a long time[1] that for, say, $U(N)$ gauge theory on the plane, Wilson loop expectations have nice limits:

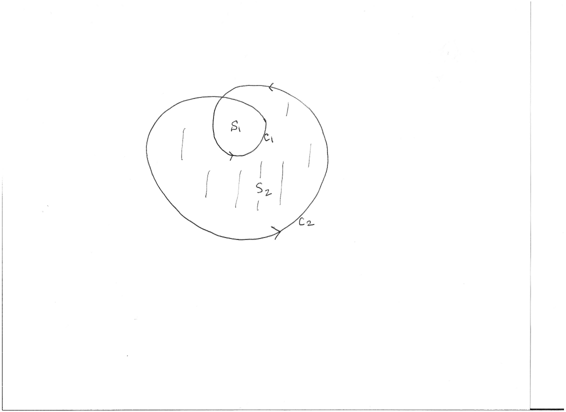
$$W_\infty(C) = \lim_{N\to\infty} \langle \mathrm{Tr}_N\big(h(C)\big)\rangle_g,$$

exists, with $g^2 N$ held fixed at some value $\tilde{g}^2$.
Note

$$\mathrm{Tr}_N = \frac{1}{N}\mathrm{Tr}$$

[1] large $T >> 20$ years

$U(N)$ Loop Example



$U(N)$/Large-N Loop Example

$$W_N(C_1 C_2) = e^{-\frac{\tilde{g}^2}{2}(S_2+2S_1)} \left(\cosh(\tilde{g}^2 S_1/N) - N \sinh(\tilde{g}^2 S_1/N)\right).$$

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So

$$W_\infty(C_1 C_2) = e^{-\frac{\tilde{g}^2}{2}(S_2+2S_1)} \left(1 - \tilde{g}^2 S_1\right).$$

The $N = \infty$ 'measure'

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Disjointly supported f give rise to independent matrices.
When $N \to \infty$ this produces a *free white noise*

Free noise

For each $f \in L^2(\mathbb{R}^2)$ then

$$F_\infty(f)$$

is a Wigner semi-circular variable, and disjointly supported f's produce mutually free variables. [S. 2008]

Thank you !