Intersection Pairings and Chern-Simons Theory on Seifert Manifolds

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August 3, 2009

Witten's Chern-Simons Invariant

- P be a (trivial) principal G fibre bundle over M. Denote the space of connections by $\mathcal{B}$ and the space of gauge transformations by $\mathcal{H}$.
- The action at level k is

$$I(B) = i\frac{k}{4\pi}\int_M \mathrm{Tr}\left(B \wedge dB + \frac{2}{3}B \wedge B \wedge B\right)$$

  and Tr is normalized so that under large gauge transformations $I(B^g) = I(B) + 2\pi i n$.
- The invariant

$$Z_{k,G}[M] = \int_{\mathcal{B}} \exp\left(I(B)\right)$$

The underlying manifolds of interest

- Σ a smooth genus g Riemann surface with ω a unit volume Kähler form on Σ.
- $M \equiv M_{(g,p)}$ the Seifert manifold that is presented as a degree $-p$, $U(1)$ bundle over Σ.
- κ a connection on M so that

$$d\kappa = p\,\pi_*(\omega), \quad \int_M \kappa \wedge d\kappa = p$$

$\Sigma \times S^1$ $(p = 0)$

- $\mathfrak{M}$ be the moduli space of flat G connections on Σ.
- $\mathfrak{L} \to \mathfrak{M}$ the fundamental line bundle whose first Chern class agrees with the natural symplectic form on $\mathfrak{M}$.
- Hirzebruch-Riemann-Roch and Kodaira vanishing tell us

$$\dim \mathrm{H}^0(\mathfrak{M}, \mathfrak{L}^k) = \int_{\mathfrak{M}} \mathrm{Todd}(\mathfrak{M}) \wedge \mathrm{Ch}(\mathfrak{L}^k)$$

- The Chern-Simons invariant, $Z_{k,G}[\Sigma \times S^1]$ is the dimension of the Hilbert space of states and thus provides us with another formulation of E. Verlinde's dimension count.

This raises a

Question

Are there other 3-manifolds whose Chern-Simons invariants, or parts of them, can be expressed as intersection pairings on an appropriate $\mathfrak{M}$?

Answer

- The question has been answered in the affirmative by Beasley and Witten for M using non-Abelian localization.
- They are able to write the portion of the Chern-Simons invariant which is localized on the smooth part of the moduli space of Yang-Mills connections as (equation (5.176) in their paper) as

$$
\begin{aligned}
Z_{k,\,G}[M]|_{\mathfrak{M}} \quad = \quad & \frac{1}{|\Gamma|} \exp\left(i\frac{\pi}{2}\eta_0\right) \int_{\mathfrak{M}} \mathrm{Todd}(\mathfrak{M}) \\
& . \ \exp\left(k\Omega(\mathfrak{M}) - i\frac{p}{2\pi}(k + c_{\mathfrak{g}})\Theta(\mathfrak{M})\right)
\end{aligned}
$$

Where we are going today?

- We will show that the Chern-Simons path integral on M is equivalent to the following path integral on Σ

$$Z_{k,G}[M,(x_i,R_i)]$$
$$= \exp\left(i\frac{\pi}{2}\eta_0\right)\int_{\mathcal{A}\times\prod_i M_{R_i}}\widehat{A}(\mathcal{A}\times\prod_i M_{R_i})$$
$$.\ \exp\left((k+c_{\mathfrak{g}})\Omega(\mathcal{A})+\sum_i\omega(M_{R_i})-i\frac{p}{2\pi}(k+c_{\mathfrak{g}})\Theta(\mathcal{A})\right)$$

The Cohomology Ring and TFT

- Witten established that the analogue of Donaldson theory on Σ can be mapped to Yang-Mills theory there.
- The TFT is designed to probe the cohomology ring of the moduli space $\implies$ Yang-Mills theory will do just that.
- Let P be a principle bundle on Σ with compact structure group G. Let $\mathcal{A}$ denote the space of connections on P and $\mathcal{G}$ the space of gauge transformations. The action of Yang-Mills theory is

$$\begin{aligned} S(F_A, \psi, \phi) &= \frac{1}{4\pi^2}\int_\Sigma \mathrm{Tr}\left(i\phi F_A + \frac{1}{2}\psi\wedge\psi\right) \\ &\quad + \frac{\epsilon}{8\pi^2}\int_\Sigma \omega\ \mathrm{Tr}\,\phi^2 \end{aligned}$$

Make CS look like the right TFT

- Use the $U(1)$ bundle structure and the associated nowhere vanishing vector field to decompose connections as

$$B = A + \kappa \frac{1}{2\pi}\phi, \quad \iota_\kappa A = 0$$

- The Chern-Simons action is now

$$I(A,\phi) = i\frac{k}{4\pi^2}\int_M \left(\pi\kappa \operatorname{Tr} A\iota_\kappa dA + \kappa \operatorname{Tr} \phi F_A + \kappa d\kappa \frac{1}{4\pi}\operatorname{Tr} \phi^2\right)$$

and this has some resemblance to the YM action.

- There are some differences and perhaps the most glaring is that the term proportional to $\psi \wedge \psi$ is absent.

$N = 1$ supersymmetric CS to the Rescue

- However, a variation on the theme is now available. 'Twist' the $N = 1$ supersymmetric Chern-Simons theory. The twisted version has action $I(B)$ augmented with

$$\frac{k}{8\pi^2}\int_M \kappa \,\mathrm{Tr}\,\psi\wedge\psi, \quad \iota_\kappa\psi = 0$$

The action becomes

$$I(A,\phi,\psi) = \frac{k}{4\pi^2}\int_M \left(i\pi\kappa\,\mathrm{Tr}\,A\iota_\kappa dA + \kappa\,\mathrm{Tr}\left(i\phi F_A + \frac{1}{2}\psi\wedge\psi\right) + \frac{i}{4\pi}\kappa d\kappa\,\mathrm{Tr}\,\phi^2\right)$$

so that now the resemblance of the two theories is rather remarkable.

- The supersymmetry transformations are closely linked to those in BW

$$QA = i\psi, \quad Q\psi = -d_A\phi - 2\pi\iota_\kappa dA$$

A Universal Bundle- Atiyah and Singer

- $G \to P \to \Sigma$ be a principle G bundle. $\mathcal{A}$ the space of connections on P and $\mathcal{G}$ the group of gauge transformations (bundle automorphisms).

$$(\mathcal{A} \times P)/\mathcal{G} = \mathcal{Q}$$

  Now G operates on $\mathcal{Q}$ and infact $\mathcal{Q}$ is itself the total space of a principle bundle (as G action commutes with $\mathcal{G}$)

$$\mathcal{Q} \to \mathcal{Q}/G = \mathcal{A}/\mathcal{G} \times \Sigma$$

- There is a natural connection on $\mathcal{Q}$ and from it we can define a curvature 2-form and then Chern classes, via Chern-Weil theory, for an associated rank n vector bundle $\mathcal{E} = \mathcal{Q} \times_G \mathbb{C}^n \to \mathcal{A}/\mathcal{G} \times \Sigma$.
- Restrict to some moduli space $\mathfrak{M} \subset \mathcal{A}/\mathcal{G}$, for which the above construction makes sense.

Chern Classes

- Decompose the curvature 2-form on $\mathcal{E}$ into its Kunneth components as
$$1 \otimes F_A + \Psi + \Phi \otimes 1 \in$$
- Fix on $G = SU(r)$ then $c_1(\mathcal{E})$ vanishes and the second Chern class decomposes as
$$\begin{aligned} & c_2(\mathcal{E}) \\ & = \frac{1}{4\pi^2} \operatorname{Tr}\left(\Phi \otimes F_A + \frac{1}{2}\Psi \wedge \Psi\right) \\ & \quad + \frac{1}{4\pi^2} \operatorname{Tr} \Phi \wedge \Psi + \frac{1}{8\pi^2} \operatorname{Tr} \Phi^2 \otimes 1 \\ & \quad\quad = \Omega(\mathcal{E}) + \gamma(\mathcal{E}) + \Theta(\mathcal{E}) \end{aligned}$$
- Make the identifications $\Phi = i\phi$ and $\Psi = \psi$, and consider these as forms on $\mathcal{A}/\mathcal{G} \times \Sigma$
$$S(F_A, \psi, \phi) = \pi_* \left(\Omega(\mathcal{E}) - \epsilon\Theta(\mathcal{E}) \otimes \omega\right) \simeq \Omega(\mathcal{A}) - \epsilon\Theta(\mathcal{A})$$
where $\pi : \mathcal{A}/\mathcal{G} \times \Sigma \to \mathcal{A}/\mathcal{G}$ is projection onto the first factor.

Reminder

- $c_2(\mathrm{End}\,\mathcal{E}) = 2rc_2(\mathcal{E})$.
- The main interest here will be on the trace free part $\mathrm{End}_0\,\mathcal{E}$. Note that the classes on $\mathcal{E}$ are in the 'fundamental' representation while they are taken to be in the adjoint representation on $\mathrm{End}_0\,\mathcal{E}$.

Grothendieck-Riemann-Roch

- The tangent bundle, $\mathrm{T}_{\mathfrak{M}}$, of $\mathfrak{M}$ is given by

$$\mathrm{T}_{\mathfrak{M}} \simeq R^1\pi_*\mathrm{End}_0\,\mathcal{E}$$

under the map $\pi : \mathfrak{M} \times \Sigma \to \mathfrak{M}$ onto the first factor. The Grothendieck-Riemann-Roch theorem states that

$$\mathrm{Ch}(\mathrm{T}_{\mathfrak{M}}) - \mathrm{Ch}(\pi_*\mathrm{End}_0\,\mathcal{E}) = -\pi_*\left(\mathrm{Ch}(\mathrm{End}_0\,\mathcal{E})(1-(g-1)\omega)\right)$$

- For the spaces that we are interested in the direct image sheaf $R^0\pi_*\mathrm{End}_0\,\mathcal{E}$ is trivial so

$$\mathrm{Ch}(\mathrm{T}_{\mathfrak{M}}) = -\pi_*\left(\mathrm{Ch}(\mathrm{End}_0\,\mathcal{E})(1-(g-1)\omega\right)$$

The $\widehat{A}$ and Todd geni

- The Pontrjagin class of the tangent bundle $P(T_{\mathfrak{M}})$ is given by

$$P(T_{\mathfrak{M}}) = \det_{\mathbf{k}}(1 + \mathrm{ad}\,\Phi/2\pi)^{2g-2} = \prod_{\mathbf{k}_+}\left(1 - \left(\frac{\alpha(\Phi)}{2\pi}\right)^2\right)^{2g-2}$$

- The Todd class of the tangent bundle of $\mathfrak{M}$ is

$$\mathrm{Todd}(\mathfrak{M}) = \exp\frac{1}{2}c_1(T_{\mathfrak{M}}) \,.\, \left(\frac{\det_{\mathbf{k}}\sin\left(\mathrm{ad}\,\Phi/4\pi\right)}{\det_{\mathbf{k}}\left(\mathrm{ad}\,\Phi/4\pi\right)}\right)^{1-g}$$

- Use GRR to determine the first Chern class of the moduli space by comparing with the theorem of Drezet and Narasimhan, that $c_1(T\mathfrak{M}) = 2r\Omega(\mathfrak{M})$ to find

$$\Omega(\mathfrak{M}) = \frac{1}{4\pi^2}\int_\Sigma\left(i\,\mathrm{Tr}\,\phi F_A + \frac{1}{2}\,\mathrm{Tr}\,\Psi\wedge\Psi\right)$$

which is the form one would expect on $F_A = 0$.

Intersection Pairings on Moduli Spaces

- With abuse of notation we denote those classes on $\mathcal{A}$ by the same symbols as those on $\mathfrak{M}$,

$$\int_{\mathcal{A}\otimes\Omega^0(\Sigma,\mathfrak{g})} \exp\left(S(F_A,\psi,\phi)\right) \equiv \int_{\mathcal{A}} \exp\left(\Omega(\mathcal{A}) - \epsilon\Theta(\mathcal{A})\right)$$

- Witten shows that this path integral essentially devolves to one on the moduli space,

$$\begin{aligned} \int_{\mathcal{A}} \mathrm{e}^{\left(\Omega(\mathcal{A}) - \epsilon\Theta(\mathcal{A})\right)} &= \int_{\mathfrak{M}} \mathrm{e}^{\left(\Omega(\mathfrak{M}) - \epsilon\Theta(\mathfrak{M})\right)} \\ &\quad + \text{terms non} - \text{analytic in } \epsilon \end{aligned}$$

and the non-analytic terms vanish as $\epsilon \to 0$ (provided $\mathfrak{M}$ is not singular). The non-analytic terms arise from higher fixed points of the action, that is, from non-flat solutions to $d_A * F_A = 0$.

Supersymmetric Quantum Mechanics: Chern-Simons to Yang-Mills

We start with the Chern-Simons theory, then integrate out modes in the bundle direction, to be left with a theory on the Riemann surface. We stop at this ‘half-way’ point where we have non-Abelian Yang-Mills theory on Σ.

- Impose the gauge condition that ϕ is constant in the fibre direction,

$$\iota_\kappa . d\phi = 0$$

- Decompose all the sections in terms of characters of the S^1 action on M

$$A = \sum_{n=-\infty}^{\infty} A_n, \quad \iota_\kappa dA_n = -2\pi i n A_n, \quad A_n \in \Omega^1(\Sigma, L^{\otimes -np} \otimes \text{ad}\, P)$$

where L is the line bundle that defines M (similarly for other sections).

Supersymmetric Quantum Mechanics continued

- Integrate out all those Fourier modes of fields such that $n \neq 0$. By the gauge condition ϕ has no such modes. Note that (with A_0 denoted by A again)

$$I(A, \phi, \psi) = kS(F_A, \psi, \phi) + \Delta I$$

where

$$\Delta I = \frac{k}{4\pi^2} \int_\Sigma \sum_{n \neq 0} \mathrm{Tr}\, (A_{-n} \wedge (2\pi n + \mathrm{ad}\, \phi/2\pi) A_{-n} + \psi_n \wedge \psi_{-n})$$

Let

$$\exp i\Gamma(\phi) = \int \prod_{n \neq 0} dA_n\, d\psi_n\, e^{\Delta I} \Delta_{\mathrm{FP}}(\phi)$$

where the Faddeev-Popov determinant $\Delta_{\mathrm{FP}}(\phi)$ arises because of the gauge choice.

SSQM continued

- The supersymmetric quantum mechanics path integral gives, for ϕ valued in the Cartan subalgebra,

$$\begin{aligned} &\exp i\Gamma(A,\phi) \\ &= \exp\left(i\frac{\pi}{2}\eta_0\right)\widehat{A}(i\phi)\wedge\exp\left(i\frac{c_{\mathbf{g}}}{4\pi^2}\int_\Sigma \mathrm{Tr}\left(\phi.F_A+\frac{p}{4\pi}\phi^2\omega\right)\right)\end{aligned}$$

- However, there are still 2 issues that we need to deal with
- Extend the result to general sections $\phi\in\Gamma(\Sigma,\mathrm{ad}\,P)$.
- Make sure that supersymmetry is preserved since at the moment the original supersymmetry at the level of the zero modes,

$$QA=i\psi,\quad Q\psi=-d_A\phi,\quad Q\phi=0,$$

is not respected.

Corrections

- The absolute value of the determinant is as before.
- Full gauge invariance is restored by a finite counter term
- We can add another finite renormalization

$$\exp \frac{c_{\mathfrak{g}}}{8\pi^2} \int_\Sigma \mathrm{Tr}\, \psi \wedge \psi$$

  to obtain a supersymmetric theory.
- The gauge invariant and supersymmetric evaluation of the path integral along the fibres of M is

$$\exp i\Gamma(A, \phi, \psi). \exp\left(-i\frac{\pi}{2}\eta_0\right)$$
$$= \widehat{A}(i\phi) \wedge \exp\left(\frac{c_{\mathfrak{g}}}{4\pi^2} \int_\Sigma \mathrm{Tr}\left(i\phi.F_A + \frac{1}{2}\psi \wedge \psi + i\frac{p}{4\pi}\phi^2\omega\right)\right)$$

So finally...

- With the identifications that $\psi \simeq \Psi$ and $\phi \simeq -i\Phi$ and as $c_{\mathfrak{g}} = r$ we have

$$\exp i\Gamma(A, \phi, \psi) = \exp\left(i\frac{\pi}{2}\eta_0\right) \mathrm{Todd}(\mathcal{A}) \exp\left(-i\frac{p}{2\pi}c_{\mathfrak{g}}\Theta(\mathcal{A})\right)$$

-

$$\begin{aligned} Z_{k,\,G}[M] &= \exp\left(i\frac{\pi}{2}\eta_0\right)\int_{\mathcal{A}} \mathrm{Todd}(\mathcal{A}) \\ &\quad .\ \exp\left(k\Omega(\mathcal{A}) - i\frac{p}{2\pi}(k + c_{\mathfrak{g}})\Theta(\mathcal{A})\right) \end{aligned}$$

- BW (with $n = -p$ because of a different choice of orientation and a slightly different normalization of Θ)

$$\begin{aligned} Z_{k,\,G}[M]|_{\mathfrak{M}} &= \frac{1}{|\Gamma|}\exp\left(i\frac{\pi}{2}\eta_0\right)\int_{\mathfrak{M}} \mathrm{Todd}(\mathfrak{M}) \\ &\quad .\ \exp\left(k\Omega(\mathfrak{M}) - i\frac{p}{2\pi}(k + c_{\mathfrak{g}})\Theta(\mathfrak{M})\right) \end{aligned}$$

Wilson Lines

A Wilson line is a combination of a knot K and an irreducible representation R of the group G,

$$W_R(K) = \mathrm{Tr}_R \, P \exp \int_K B$$

Since our manifold is a S^1 fibration there is a special class of knots which are located at point $x \in \Sigma$ on the base of the fibration and which run along the fibre,

$$W_R(x) = \mathrm{Tr}_R \, P \exp \left(\int_{S^1} \kappa\phi/2\pi \right) = \mathrm{Tr}_R \exp \left(\phi(x)/2\pi\right)$$

$$Z_{k,G}[M, (x_i, R_i)] = \int_{\mathcal{B}} \exp\left(I(B)\right) \prod_{i=1} W_{R_i}(x_i)$$

Co-adjoint orbits

A geometric way in which to add such traces is through the introduction of co-adjoint orbits. Let $\lambda \in \mathfrak{g}^*$ ($\mathfrak{g}^*$ is the dual of $\mathfrak{g}$, however, we identify the two so that an invariant inner product $< f, \phi > \equiv \mathrm{Tr}\, f\phi$, $f \in \mathfrak{g}^*$, $\phi \in \mathfrak{g}$) then the orbit through λ is

$$M_\lambda = \left\{ g^{-1} \lambda g;\ \forall g \in G \right\}$$

while the stabilizer of λ is

$$G(\lambda) = \left\{ g \in G :\ g^{-1} \lambda g = \lambda \right\}$$

If λ is regular ($\det_{\mathbf{k}} (\mathrm{ad}\, \lambda) \neq 0$) then $G(\lambda) = T$ and we consider this case for now so that $M_\lambda = G/G(\lambda) = G/T$.

Quantization

The homogeneous space G/T comes equipped with a natural G invariant symplectic 2-form (the Kirillov-Konstant form) Ω_λ given by

$$\Omega_\lambda(X, Y) =< \lambda, [X, Y] >= \mathrm{Tr}\,(\lambda\,[X, Y]) \quad X,\ Y \in \mathfrak{g}$$

Kirillov tells us that for $\lambda = \Lambda + \rho$ regular, Λ an element of the weight lattice and ρ the Weyl vector then

$$\mathrm{Tr}_\lambda\,(\exp\phi/2\pi) = j_{\mathfrak{g}}^{-1/2}(\phi/2\pi) \int_{M_\lambda} \exp\left(i\frac{1}{2\pi} < \lambda, \phi > +\Omega_\lambda \right)$$

Now we see that geometrically we should product in the co-adjoint orbits so consider the space $\mathcal{B} \times \prod_i M_{R_i}$, and we have

$$Z_{k,G}[M, (x_i, R_i)] = \int_{\mathcal{B}\times\prod_i M_{R_i}} \exp\left(I(B)\right) \prod_{i=1} j_{\mathfrak{g}}^{-1/2}(\phi(x_i)/2\pi)\, \exp\omega(M_{R_i})$$

where, in analogy with $\Omega(\mathcal{A})$,

$$\omega(M_{R_i}) = \frac{i}{2\pi}\,\mathrm{Tr}\,\lambda_i\phi(x_i) + \Omega_{R_i}$$

Finally, finally..

We have the following:

Lemma

(Lemma 8.5 [Berline-Getzler-Vergne]) The equivariant $\widehat{A}$-genus, $\widehat{A}_{\mathbf{g}}(X, G/T)$, of the Riemannian manifold G/T and $j_{\mathbf{g}}^{-1/2}(X)$ represent the same class in equivariant deRham cohomology.

Consequently

$$
\begin{aligned}
& Z_{k,G}[M,(x_i,R_i)] \\
& = \exp\left(i\frac{\pi}{2}\eta_0\right) \int_{\mathcal{A}\times\prod_i M_{R_i}} \widehat{A}(\mathcal{A}\times\prod_i M_{R_i}) \\
& . \exp\left((k+c_{\mathfrak{g}})\Omega(\mathcal{A}) + \sum_i \omega(M_{R_i}) - i\frac{p}{2\pi}(k+c_{\mathfrak{g}})\Theta(\mathcal{A})\right)
\end{aligned}
$$