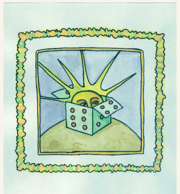
Partial Hypoellipticity of Differential Operators

Tsegaye Gedif Ayele (PhD)
Department of Mathematics
Addis Ababa University- Ethiopia
Email:tsegayeg@math.aau.edu.et

February 15, 2008

Bonn - Germany



Notations and Basic Definitions

- *Support* of function $\varphi \equiv \text{supp}(\varphi) := \overline{\{x \in \mathbb{R}^n : \varphi(x) \neq 0\}}$
- $D(\mathbb{R}^n) \equiv C_0^\infty(\mathbb{R}^n) := \{\varphi \in C^\infty(R^n) : supp(\varphi) \subset\subset \mathbb{R}^n\}$
- $S(\mathbb{R}^n) := \{\varphi \in C^\infty(\mathbb{R}^n) : D^\beta \varphi \to 0 \ (x \to \infty) \qquad \forall \beta \in \mathbb{N}_0^n\}$
- $(f, \varphi) = F[\varphi] \qquad \forall \varphi \in D$ - *linear, continuous functional.*
- $(f, \varphi) = F[\varphi] \qquad \forall \varphi \in S$ - *linear, continuous functional.*
- $D' = \{(f, \varphi) = F[\varphi]\}$ and $S' = \{(f, \varphi) = F[\varphi]\}$
- All members of D and S are called basic functions and all members of D' and S' are called distributions. And moreover,

$$D \subset S \subset S' \subset D'$$

Definition 1 *A distribution on a non-empty open set* $\Omega \subset \mathbb{R}^n$ *is any continuous linear functional on the space of basic functions.*

We will write the value of the functional (distribution) f on the basic function φ as $(f, \varphi) = F[\varphi]$.

- A distribution f is a functional on the space of basic functions, that is, with each basic function φ there is associated a (complex-valued) number $(f, \varphi = F[\varphi])$.
- A distribution f is a linear functional on the space of basic functions, that is if φ and ψ are basic functions and λ and μ are complex numbers, then

$$F[\lambda\varphi + \mu\psi] := (f, \lambda\varphi + \mu\psi) = \lambda F[\varphi] + \mu F[\psi] \qquad (1)$$

- A distribution is a continuous functional on the space of basic functions, that is if $\varphi_k \to \varphi$ in the space of basic functions as $k \to \infty$, then $(f, \varphi_k) \to (f, \varphi)$

Definition 2 *A generalized derivative $D^\alpha f$ of a distribution f is defined as:*

$$(D^\alpha f, \varphi) := (-1)^{|\alpha|}(f, D^\alpha \varphi) \tag{2}$$

for any test function φ.

Distributions have derivatives of all order. Even locally Lebesgue integrable functions that are discontinuous are infinitely differentiable in distributional sense.

Definition 3 *Fourier transforms and convolutions in S.*

- *Let $\varphi \in S$, then*

$$\mathcal{F}\varphi(\xi) := (2\pi)^{-\frac{n}{2}} \int_{R^n} e^{-i<x,\xi>} \varphi(x)dx \tag{3}$$

and

$$\mathcal{F}^{-1}\varphi(\xi) := (2\pi)^{-\frac{n}{2}} \int_{R^n} e^{i<x,\xi>} \varphi(x)dx \tag{4}$$

- *The operator of inversion $\mathcal{I} : S \longrightarrow S$ operating by the formula $\mathcal{I}\varphi(x) = \varphi(-x)$ is linear and continuous. Moreover,*

$$\mathcal{F}^{-1}\varphi(\xi) = \mathcal{F}\varphi(-\xi) = \mathcal{F}\mathcal{I}\varphi(\xi).$$

That is

$$\mathcal{F}\mathcal{F}^{-1}\varphi(\xi) = \mathcal{F}\mathcal{F}\mathcal{I}\varphi(\xi) \Leftrightarrow \mathcal{F}\mathcal{F}\mathcal{I} = I$$

Thus

$$\mathcal{F}\mathcal{F} = \mathcal{I}$$

and consequently

$$\mathcal{F}\mathcal{F}\varphi(x) = \varphi(-x) \qquad \forall\varphi \in S.$$

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- If $\varphi, \psi \in S$, then the *convolution* $\varphi * \psi \in S$ and

$$(\varphi * \psi)(x) := (2\pi)^{-\frac{n}{2}} \int \varphi(x-y)\psi(y)dy$$

$$= (2\pi)^{-\frac{n}{2}} \int \varphi(y)\psi(x-y)dy = (\psi * \varphi)(x) \qquad (5)$$

- If the Translation $T_h\varphi(y) = \varphi(h+y)$ and $T_h\varphi(-y) = \varphi(h-y)$, then

$$T_x\mathcal{I}\varphi(y) = T_x\varphi(-y) = \varphi(x-y)$$

- If the Translation $T_h\varphi(y) = \varphi(h+y)$ and $T_h\varphi(-y) = \varphi(h-y)$, then

$$T_x\mathcal{I}\varphi(y) = T_x\varphi(-y) = \varphi(x-y)$$

- From the above follows that

$$(\varphi * \psi)(x) = (2\pi)^{-\frac{n}{2}} \int \varphi(y)(\mathcal{I}T_x)\psi(y)dy$$

$$= (2\pi)^{-\frac{n}{2}} \int \psi(y)(\mathcal{I}T_x)\varphi(y)dy = (\psi * \varphi)(x)$$

Definition 4 *Let $g \in S'$ and $\varphi, \psi \in S$, then holds true the following:*

- *The Fourier transform $\mathcal{F}g$ of g is given by*

$$(\mathcal{F}g, \varphi) = (g, \mathcal{F}\varphi) \qquad \forall \varphi \in S. \tag{6}$$

- *The convolution $g * \psi$ of the functions g and ψ is the quantity*

$$(g * \psi, \varphi)(x) := (g, \mathcal{I}\psi * \varphi) \qquad \forall \varphi, \psi \in S \tag{7}$$

  $g \in S'$ is operating w.r.t. y on the basic function $(\mathcal{I}T_x)\psi(y)$.

- The convolution $g * \varphi$ has the following properties:
  - It is well defined, moreover for any classical functions it coincides with classical convolution.
  - The convolution is commutative with translation. That is

$$T_h g * \varphi = T_h(g * \varphi) = (g * T_h \varphi).$$

    $G(x) = (g * \varphi)(x)$ is a classical (regular) function defined $\forall x \in \mathbb{R}^n$, infinitely differentiable and of slow growth. Moreover, $D^\alpha G(x) = (g * D^\alpha \varphi)(x) = (D^\alpha g * \varphi)(x)$.
  - $(g * \varphi, \psi) = (g, \mathcal{I}\varphi * \psi) \qquad \forall g \in S'$ and $\varphi, \psi \in S$
  - $(\mathcal{F}(g * \varphi), \psi) = (\mathcal{F}g\mathcal{F}\varphi, \psi)$
  - $(g * \varphi) * \psi = g * (\varphi * \psi)$

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- A generalized function u is a sum of point sources

$$u(x) = (u * \delta)(x)$$

- In view of $P(D)\mathcal{E}(x) = \delta(x)$,

$$u(x) = (u * \delta)(x) = (u * P(D)\mathcal{E})(x) = P(D)(u * \mathcal{E})(x)$$
$$= (P(D)u * \mathcal{E})(x) = (f * \mathcal{E})(x) = (\mathcal{E} * f)(x)$$

- For this reason $u(x) = (\mathcal{E} * f)(x)$

Statement of the Problem

Consider a nonzero linear differential operator

$$P(D) = \sum_{\alpha \in K} a_\alpha D^\alpha \tag{8}$$

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Where K is a finite set in $\mathbb{N}_0^n$, $a_\alpha \in K$ are constants and for $\alpha = (\alpha_1, ..., \alpha_n) \in K$, $|\alpha| = \alpha_1 + ... + \alpha_n$, $\max|\alpha| = m$

$$D^\alpha = D_1^{\alpha_1} \times ... \times D_n^{\alpha_n} \quad \text{and} \quad D_j^{\alpha_j} = \frac{1}{i}\frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}}$$

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Aim: To describe the condition of partial hypoellipticity of operator (1) in terms of its Fundamental Solutions.

Discussion of the result

Definition 5 *A distribution $\mathcal{E}$ is called a fundamental solution for differential operator (1) if, and only ,if*

$$P(D)\mathcal{E} = \delta \tag{9}$$

where δ is the Dirac delta function.

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The existence of a fundamental solution $\mathcal{E}$ in D' of any differential operator of the type (1) was proved by B. Malgrange (1953) and L. Ehrenpresis (1954) independently, whereas its existence in S' was proved by L. Hormander (1958).

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Equation (9) in the class S' is equivalent to the algebraic equation

$$\mathcal{P}(i\xi)\mathcal{F}\mathcal{E} = (2\pi)^{-\frac{n}{2}} \tag{10}$$

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$$\mathcal{P}(\xi)u = f \tag{11}$$

where $\mathcal{P}$ is a nonzero polynomial of degree m and f is a specified generalized function in S'.

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where $\mathcal{P}$ is a nonzero polynomial of degree m and f is a specified generalized function in S'.

The solvability of the problem of "*division*" was proved in 1958 by Hormander and Lojasiewicz and consequently the existence of a fundamental solution $\mathcal{E}$ from S'.

Lemma 1 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set,* $u \in D'(\Omega)$ *and* $\psi \in D(\mathbb{R}^n)$*. Then the set*

$$\Omega_\psi = \bigcap_{y \in supp(\psi)} (\Omega + y)$$

is open. Moreover, the convolution $u * \psi$ *exists in* Ω_ψ*.*
That is , $\forall \varphi \in D(\Omega_\psi)$,

$$(u * \psi, \varphi)(x) := (u(x), (\psi(y), \varphi(x + y)))$$

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Definition 6 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and let* $u \in D'(\Omega)$ *a differential operator (1) is called elliptic, if* $P(D)u \in \mathcal{A}(\Omega)$*, then* $u \in \mathcal{A}(\Omega)$*.*

Definition 7 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and let* $u \in D'(\Omega)$ *a differential operator (1) is called hypoelliptic, if* $P(D)u \in C^\infty(\Omega)$*, then* $u \in C^\infty(\Omega)$*.*

Definition 7 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and let* $u \in D'(\Omega)$ *a differential operator (1) is called hypoelliptic, if* $P(D)u \in C^\infty(\Omega)$*, then* $u \in C^\infty(\Omega)$*.*

Lemma 2 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set,* $m, n \in \mathbb{Z}$ *are such that* $0 \leq m < n$ *and* $x = (x', x'')$ *for* $x' \in \mathbb{R}^m,\ x'' \in \mathbb{R}^{n-m}$*. Let*

$\forall \psi \in D(\mathbb{R}^m),\ \psi_m = \psi(x') \times \delta(x'') : supp(\psi_m) = supp(\psi) \times \{0\},$

then the set:

$$\Omega_{\psi_m} = \bigcap_{y \in supp(\psi_m)} (\Omega + y)$$

is open.

Lemma 3 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and* $u \in D'(\Omega)$ *then the convolution* $u * \psi_m$ *exists in* Ω_{ψ_m}, *that is,* $\forall \varphi \in D(\Omega_{\psi_m})$

$$(u * \psi_m, \varphi)(x) := (u(x), (\psi(y'), \varphi(x + y')))$$

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Definition 8 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and* $u \in D'(\Omega)$ *a differential operator (1) is called partially hypoelliptic with respect to the plane* $x'' = 0$, *if* $P(D)u \in C^\infty(\Omega)$, *then for any function* $\psi \in D(\mathbb{R}^m)$ *the convolution* $u * \psi_m \in C^\infty(\Omega_{\psi_m})$.

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Remark 1 *In definition(8), if* $m = 0$*, then* $\psi_0 = \delta$ *and in this case we obtain definition (7).*

Theorem 1 *For the differential operator (1) to be partially hypoelliptic with respect to the plane* $x'' = 0$, *it is necessary and sufficient that there exists a fundamental solution* $\mathcal{E}$ *of operator (1) such that for any function* $\psi \in D(R^m)$ *the convolution* $\mathcal{E} * \psi_m \in C^\infty(\mathbb{R}^n \backslash supp(\psi_m))$.

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Remark 2 *From the existence of such fundamental solution follows that every fundamental solution satisfies*

$$\mathcal{E} * \psi_m \in C^\infty(\mathbb{R}^n \setminus supp(\psi_m)).$$

Remark 3 *If* $m = 0$*, then* $\psi_0 = \delta$ *and we obtain Hormander's theorem about description of hypoellipticity of operator (1) in terms of fundamental solutions which is stated as follows:*

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Theorem 2 *For operator (1) to be hypoelliptic it is necessary and sufficient that there exists a fundamental solution $\mathcal{E}$ of operator (1) such that $\mathcal{E} \in C^{\infty}(\mathbb{R}^n \backslash \{0\})$*

Remark 3 *If* $m = 0$, *then* $\psi_0 = \delta$ *and we obtain Hormander's theorem about description of hypoellipticity of operator (1) in terms of fundamental solutions which is stated as follows:*

Theorem 2 *For operator (1) to be hypoelliptic it is necessary and sufficient that there exists a fundamental solution* $\mathcal{E}$ *of operator (1) such that* $\mathcal{E} \in C^\infty(\mathbb{R}^n \backslash \{0\})$

Remark 4 *From the existence of such fundamental solution it follows that any fundamental solution* $\mathcal{E} \in C^\infty(\mathbb{R}^n \backslash \{0\})$.

Lemma 4 *Let $\Omega \subset \mathbb{R}^n$ be an open set and F compact, then the set G such that*

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Corollary 1 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and* $\psi \in D(\mathbb{R}^n)$*, the set*

$$\Omega_\psi = \bigcap_{y \in supp(\psi)} (\Omega + y)$$

is open.

Corollary 2 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and* $\psi \in D(\mathbb{R}^m)$ *for* $m, n \in \mathbb{Z}$ *and such that* $0 \leq m < n$ *the set*

$$\Omega_{\psi_m} = \bigcap_{y \in supp(\psi_m)} (\Omega + y)$$

is open.

Note that the convolution $u * \psi$ is given by

$$(u * \psi, \varphi)(x) := (u(x), (\psi(y), \varphi(x + y))) \quad \forall \varphi \in D(\Omega_\psi)$$

Now let us show that

$$(\psi(y), \varphi(x + y)) \in D(\Omega) \quad \forall \varphi \in D(\Omega_\psi).$$

To show this, denote by

$$\Lambda_1(x) = \int \psi(y)\varphi(x+y)dy$$

and consider the following propositions.

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and consider the following propositions.

Proposition 1

$$supp(\Lambda_1) \subset supp(\varphi) - supp(\psi)$$

Proposition 2

$$If \;\; supp(\varphi) \subset \Omega_\psi, \;\; then \;\; supp(\varphi) - supp(\psi) \subset \Omega$$

Consequently, propositions (1) and (2) show that,

$$(\psi(y), \varphi(x+y)) \in D(\Omega) \quad \forall \varphi \in \Omega_\psi$$

Remark 5 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and* $u \in D'(\Omega)$, *then the convolution* $u * \psi$ *is defined on* Ω_ψ *and corollary (1) together with propositions (1) and (2) prove Lemma (1).*

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Definition 9 *Let* $\Omega \subset \mathbb{R}^n$ *be open an open set,* $u \in D'(\Omega)$ *and* $\psi \in D(\mathbb{R}^m)$, *then* $\forall \varphi \in D(\Omega_{\psi_m})$ *the convolution* $u * \psi_m$ *is given by*

$$(u * \psi_m, \varphi) := (u * (\psi \times \delta), \varphi)$$
$$= (u(x), (\psi(y'), \eta(y')\varphi(x + y')))$$
$$= (u(x), (\psi(y'), \varphi(x + y')))$$

And in similar way as above we can show that,

$$(\psi(y'), \varphi(x+y')) \in D(\Omega) \qquad \forall \varphi \in D(\Omega_{\psi_m})$$

by assuming

$$\Lambda_2(x) = \int \psi(y')\varphi(x+y')dy'$$

and making use of the following propositions:

Proposition 3

$$supp(\Lambda_2) \subset supp(\varphi) - supp(\psi_m)$$

And in similar way as above we can show that,

$$(\psi(y'), \varphi(x+y')) \in D(\Omega) \qquad \forall \varphi \in D(\Omega_{\psi_m})$$

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and making use of the following propositions:

Proposition 3

$$supp(\Lambda_2) \subset supp(\varphi) - supp(\psi_m)$$

Proposition 4

$$\textit{If } supp(\varphi) \subset \Omega_{\psi_m}, \ \textit{then } supp(\varphi) - supp(\psi_m) \subset \Omega$$

Consequently, $(\psi(y'), \varphi(x+y')) \in D(\Omega) \qquad \forall \varphi \in D(\Omega_{\psi_m})$

Remark 6 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set* $u \in D'(\Omega)$ *and* $\psi \in \mathbb{R}^m$ *for* $m, n \in Z$ *and such that* $0 \leq m < n$*, then the convolution* $u * \psi_m$ *is defined on* Ω_{ψ_m} *and corollary (2), propositions (3) and (4) prove Lemmas (2) and (3).*

Remark 6 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set* $u \in D'(\Omega)$ *and* $\psi \in \mathbb{R}^m$ *for* $m, n \in Z$ *and such that* $0 \leq m < n$, *then the convolution* $u * \psi_m$ *is defined on* Ω_{ψ_m} *and corollary (2), propositions (3) and (4) prove Lemmas (2) and (3).*

To prove Theorem (1), we need the following:

Lemma 5 *Let* $G \subset \mathbb{R}^n$ *be a closed set,* $f \in D'(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \backslash G)$ *and* $g \in \mathcal{E}'(\mathbb{R}^n)$, *then* $\quad f * g \in C^\infty(\mathbb{R}^n \backslash (supp(g) + G))$. *Moreover,* $\forall \gamma > 0$ *on* $R^n \backslash (\overline{(supp(g)^\gamma)} + G)$,

$$(f * g)(x) = ((g(y), \eta(y) f(x - y)). \qquad \forall \eta \in D(R^n): \; \eta(x) = 1$$

in some neighborhood of $supp(g)$ *and* $supp(\eta) \subset (supp(g))^\gamma$.

Remark 7 *If* $\Omega = G^c$, *then, we observe that the convolution*

$$(f * g)(x) = (g(y), \eta(y) f(x - y)) \in C^\infty(\Omega_g)$$

where the set

$$\Omega_g = (G + supp(g))^c = \mathbb{R}^n \backslash (supp(g) + G).$$

Remark 8 *In Lemma (5) if* $G = \{o\}$, *then it can be easily shown that the convolution*

$$(f * g)(x) = (g(y), \eta(y) f(x - y)) \in C^\infty(\mathbb{R}^n \backslash supp(g)).$$

(See Vladimirov [1]).

Remark 9 *Let $G \subset \mathbb{R}^n$ be a closed set,*

$$f \in D'(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \backslash G)$$

and

$$g_1, g_2, ..., g_n \in \mathcal{E}'(\mathbb{R}^n),$$

then

$$(f * g_1 * g_2 * g_3 * ... * g_n)(x) \quad exists \quad in \quad D'(\mathbb{R}^n)$$

and

$$(f * g_1 * g_2 * g_3 * ... * g_n)(x) \in C^\infty(K)$$

where

$$K = (\mathbb{R}^n \backslash (G + supp(g_1) + ... + supp(g_n))).$$

Proof of theorem(1) ► *Necessity*: Let a differential operator (1) be partially hypoelliptic with respect to the plane $x'' = 0$ and $\mathcal{E} \in D'$ be its fundamental solution, i.e.

$$P(D)\mathcal{E} = \delta,$$

then using definition (8) Lemma (5)and remarks (7) and (8) we deduce that for any $\psi \in D(\mathbb{R}^m)$ and $\psi_m(x) = \psi(x') \times \delta(x'')$ the convolution

$$(\mathcal{E} * \psi_m)(x) \in C^\infty(\mathbb{R}^n \backslash supp(\psi_m)).$$

Sufficiency: Now let $\mathcal{E} \in D'$ be a fundamental solution of operator (1)and the convolution

$$(\mathcal{E} * \psi_m)(x) \in C^\infty(\mathbb{R}^n \backslash supp(\psi_m)).$$

And let $u \in D'(\Omega)$ be a solution for the equation

$$P(D)u = f, \qquad f \in C^\infty(\Omega).$$

Then

$$\eta u * \psi_m \in C^\infty(\Omega_{\psi_m}) \quad where \quad \eta \in D(\Omega), \quad \eta(x) = 1 \quad \forall x \in \Omega' \subset\subset \Omega.$$

Indeed

$$\eta u * \psi_m = \eta u * (\psi_m * \delta) = \eta u * (\psi_m * P(D)\mathcal{E}) = \eta u * P(D)(\psi_m * \mathcal{E})$$

$$= P(D)(\eta u * (\mathcal{E} * \psi_m)) = P(D)(\eta u) * (\mathcal{E} * \psi_m) = (\eta f + f_1) * (\mathcal{E} * \psi_m)$$

where

$$\eta f \in D(\Omega), \quad f_1 \in D'(\mathbb{R}^n) : supp(f_1) \subset supp(\eta) \backslash \Omega'.$$

Since $\eta f \in D(\Omega)$, the convolution $\eta f * (\mathcal{E} * \psi_m) \in C^\infty(\mathbb{R}^n)$.

Further note that,

$$\mathcal{E} * \psi_m \in D'(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \backslash supp(\psi_m))$$

and

$$f_1 \in D'(\mathbb{R}^n) : supp(f_1) \subset supp(\eta) \backslash \Omega'.$$

Then by lemma (5)

$$f_1 * (\mathcal{E} * \psi_m) = (\mathcal{E} * \psi_m) * f_1 \in C^\infty(\mathbb{R}^n \backslash (supp(\psi_m) + supp(f_1))$$

and since

$$\mathbb{R}^n \backslash (\Omega'^c + supp(\psi_m)) \subset \mathbb{R}^n \backslash (supp(\psi_m) + supp(f_1)),$$

it follows that

$$(\mathcal{E} * \psi_m) * f_1 \in C^\infty(\mathbb{R}^n \backslash (\Omega'^c + supp(\psi_m)).$$

But

$$\Omega'^c + supp(\psi_m) = \bigcup_{y \in supp(\psi_m)} (\Omega'^c + y)$$

$$= \bigcup (\Omega' + y)^c = (\bigcap_{y \in supp(\psi_m)} (\Omega' + y))^c = \Omega'^c_{\psi_m}.$$

Consequently,

$$(\mathcal{E} * \psi_m) * f_1 \in C^\infty(\Omega'_{\psi_m})$$

and thus

$$\eta u * \psi_m) \in C^\infty(\Omega'_{\psi_m})$$

and since

$$\Omega'_{\psi_m} \subset\subset \Omega_{\psi_m}, \quad \eta u * \psi_m \in C^\infty(\Omega_{\psi_m}).$$

Next, let $x_0 \in \Omega_{\psi_m}$, then $\exists \delta > 0 : \mathcal{B}(x_0, \delta) \subset \Omega_{\psi_m}$ and

$$\Omega'_{x_0} = \mathcal{B}(x_0, \delta) - supp(\psi_m) \quad where\ \Omega'_{x_0} \subset\subset \Omega.$$

Let

$$\eta_{x_0} \in D(\Omega) : \eta_{x_0}(x) = 1 \quad \forall x \in \Omega'_{x_0}$$

and let us show that on $\mathcal{B}(x_0, \delta) \quad \eta_{x_0} u * \psi_m = u * \psi_m$. That is

$$(\eta_{x_0} u * \psi_m, \varphi) = (u * \psi_m, \varphi) \quad \forall \varphi \in D(\mathcal{B}(x_0, \delta)).$$

Indeed

$$(\eta_{x_0} u * \psi_m, \varphi) = (\eta_{x_o} u(x), (\psi_m(y), \varphi(x + y))$$

$$= (\eta_{x_o}(x) u(x), (\psi(y') \times \delta(y''), \varphi(x' + y', x'' + y'')))$$

$$= (\eta_{x_o}(x) u(x), (\psi(y'), \varphi(x' + y', x'')))$$

$$= (u(x), (\eta_{x_o}(x)\psi(y'), \varphi(x, y'))) = (u(x), \int_{R^m} \eta_{x_o}(x)\psi(y')\varphi(x, y')dy').$$

$$\forall y' \in supp(\psi), \forall x : (x + y') \in supp(\varphi) \subset \mathcal{B}(x_0, \delta),$$

the function $\eta_{x_0}(x) = 1$, because

$$x = (x', x'') = (x' + y', x'') - (y', 0) \in \mathcal{B}(x_0, \delta) - supp(\psi_m).$$

Consequently

$$(\eta_{x_0} u * \psi_m, \varphi) = (u(x), \int_{R^m} \varphi(x, y')\psi(y')dy' = (u * \psi_m, \varphi)$$

which means

$$(\eta_{x_0} u * \psi_m, \varphi) = (u * \psi_m, \varphi) \quad \forall \varphi \in \mathcal{B}(x_0, \delta). \blacktriangleleft$$

Summary

Definition	In terms of Poly.	In terms of FS.
Differential Operator (1) is *elliptic* if and only if $u \in D'(\Omega),\ P(D)u \in \mathcal{A}(\Omega) \implies u \in \mathcal{A}(\Omega)$.	For $\xi \neq 0$, the principal part $\mathcal{P}_m(\xi) \neq 0$	Any Fundamental solution $\mathcal{E}$ is such that $\mathcal{E} \in \mathcal{A}([\{0\}]^c)$
Differential Operator (1) is *hypoelliptic* if and only if $u \in D'(\Omega), P(D)u \in C^\infty(\Omega) \implies u \in C^\infty(\Omega)$.	$\frac{\mathcal{P}^{(\alpha)}(\xi)}{\mathcal{P}(\xi)} \to 0\ (\xi \to \infty)$	Any fundamental solution $\mathcal{E}$ is such that $\mathcal{E} \in C^\infty([\{0\}]^c)$
Differential Operator (1) is *partially hypoelliptic* iff $u \in D'(\Omega),\ \psi \in D(\mathbb{R}^m)$, $P(D)u \in C^\infty(\Omega) \implies u * \psi_m \in C^\infty(\Omega_{\psi_m})$.	$\frac{\mathcal{P}^{(\alpha)}(\xi)}{\mathcal{P}(\xi)} \to 0\ (\xi'' \to \infty)$ whereas ξ' remains bounded	For $m, n \in \mathbb{Z}$: $0 \leq m < n$ and $\forall\ \psi \in D(\mathbb{R}^m)$, $\mathcal{E} * \psi_m \in C^\infty([supp(\psi_m)]^c)$

References

[1] Vladimirov V. S., *Generalized Functions in Mathematical Physics. Mir publisher. Moscow, 1979.*

[2] Hormander L., *Linear Partial Differential Operators. New York, 1969.*

[3] Tsegaye G. Ayele , *About Smoothness of solutions and conditions of localization for spectral decomposition of operators with constant coefficients. Ph.D. Thesis. Moscow, 1999.*

THE END

THANK YOU