Thin tubes in mathematical physics, global analysis and spectral geometry

Daniel Grieser

February 14, 2008

Overview

1. Setup

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2. Problems in MP, SG, GA

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3. Thin tubes
4. Methods: ΨDO or no?
5. Results: Fat graphs
   - Quantum graphs
   - Expectations
   - Fat graph theorem

Overview

References

Survey:

D. Grieser. *Thin tubes in mathematical physics, global analysis and spectral geometry.* To appear in: 'Analysis on Graphs and its Applications', Symp. in Pure Math., AMS, 2008. (Also on http://www.mathematik.uni-oldenburg.de/personen/grieser/)

Details:

D. Grieser. *Spectra of graph neighborhoods and scattering.* Preprint arXiv:0710.3405, 2007.

D. Grieser and D. Jerison. *Asymptotics of eigenfunctions on plane domains.* Preprint, arXiv:0710.3665, 2007

For global analysis: Hassell-Mazzeo-Melrose, Cappell-Lee-Miller, W.Müller, J.Müller-W.Müller, Park-Wojciechowski

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Scaling

$$\lambda^{(cM)} = c^{-2}\lambda^{(M)} \qquad (c > 0)$$

(cM = all lengths are multiplied by c)

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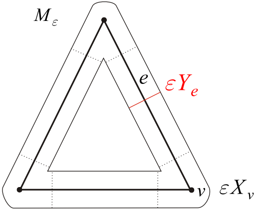
Fat graph M_ε, thickness $\varepsilon > 0$



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(GA) (Global Analysis)
Gluing formula for analytic torsion:

$$\tau(X_\ell \cup_Y X_r) = \tau(X_\ell) + \tau(X_r) - ?(Y)$$

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(GA) Decompose space into simple parts

Common theme: Thin tubes

A thin tube is...

an n-dimensional compact space M_ε of size

$\approx \varepsilon$ in $n-1$ directions and

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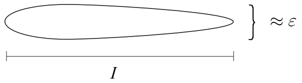

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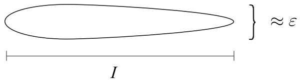

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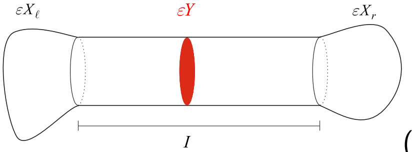

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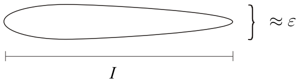

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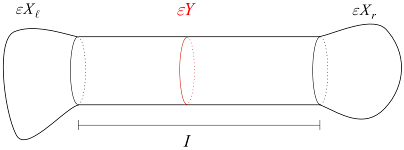

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- Matched asymptotic expansions — rigid geometry, sharp
- Matching of scattering solutions — simple, rigid geometry, very sharp
- Resolvent construction using adapted ΨDO calculus
  NOT simple, less rigid, sharp

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- Surgery calculus, $Q = Q(x, y, x', y'; z; \varepsilon)$: x, x', $p - p'$, z, ε

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Decoupling: $A_v = \{0\}$: $w = 0$ at v, no relation between edges hitting v.

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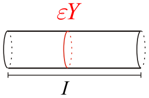
Cylinder $\mathrm{Cyl} = \varepsilon Y \times I$ (Dirichlet BC)



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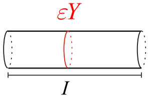

$$\Delta_{\mathrm{Cyl}} = \varepsilon^{-2}\Delta_Y + (-\partial_x^2)_I$$

$$\lambda_{k,\varepsilon}^{\mathrm{Cyl}} = \varepsilon^{-2}\nu + k^2\pi^2|I|^{-2}$$

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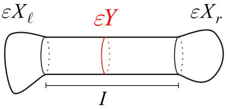


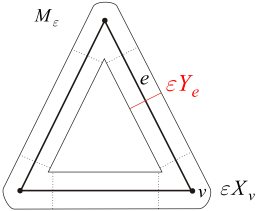
$$\Delta_{M_\varepsilon} = \varepsilon^{-2}\Delta_Y + (-\partial_x^2)_I$$

$$\lambda_{k,\varepsilon} \overset{?}{=} \varepsilon^{-2}\nu + k^2\pi^2|I|^{-2} + O(\varepsilon)$$

$$u_{k,\varepsilon}(x,y) \overset{?}{=} \varphi(y/\varepsilon) \cdot \sin(k\pi|I|^{-1}x) + O(\varepsilon)$$

(Perturbation is of same order as $|I| \rightsquigarrow |I| + \varepsilon$)

Expectations: Fat graphs



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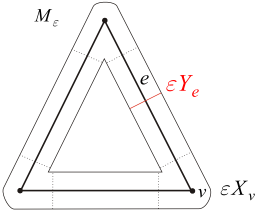

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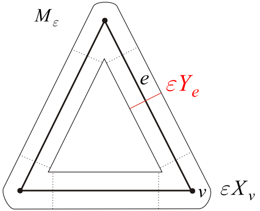

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Therefore:

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$\mu_k =$ eigenvalues of $-\partial_x^2$ on edges, with corner conditions.

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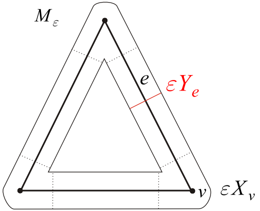

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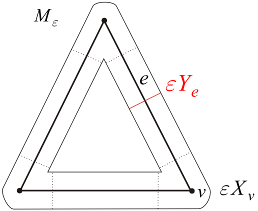

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- If yes, which corner conditions?

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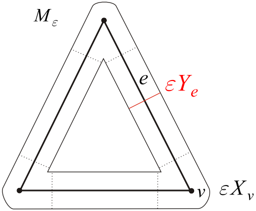

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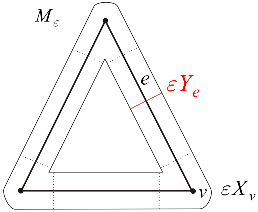

$$\downarrow \varepsilon \to 0$$

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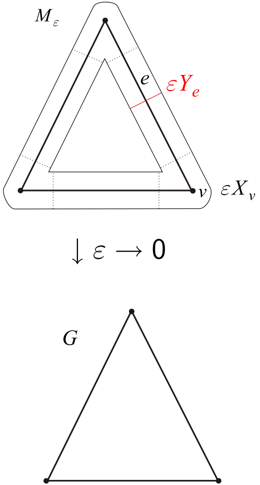

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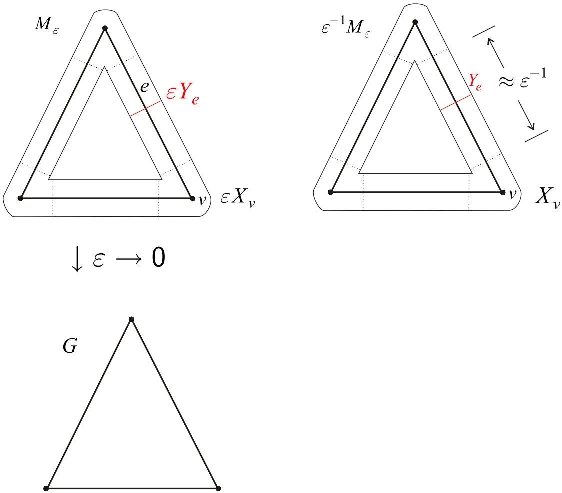

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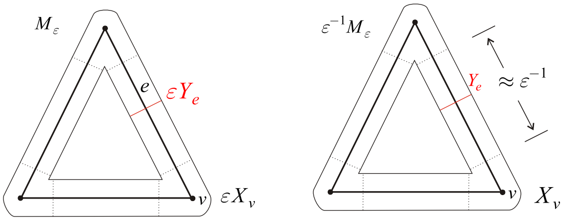

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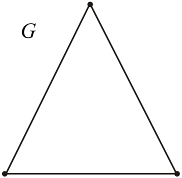

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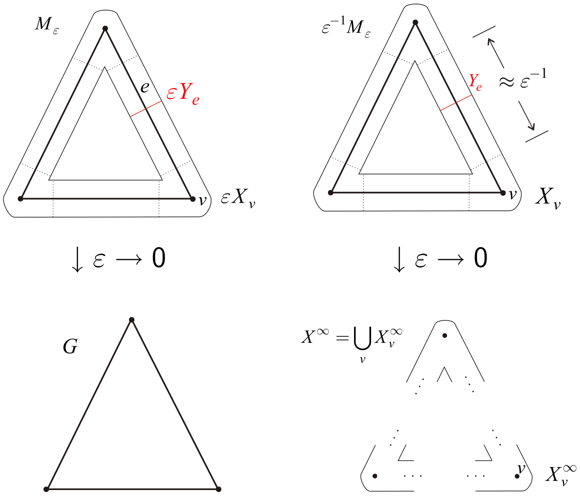

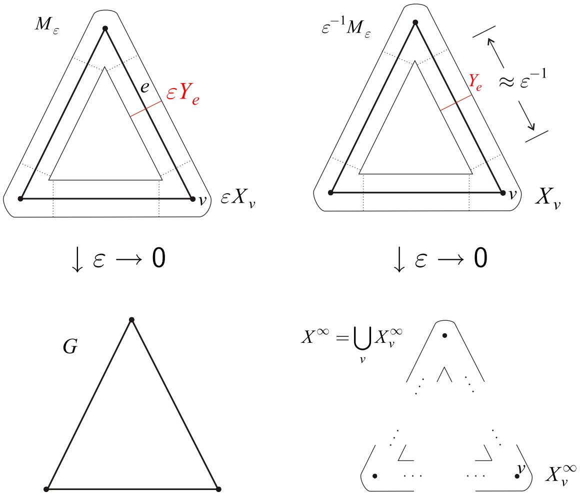

Eigenvalues: $\lambda_{k,\varepsilon}$ $\varepsilon^2 \lambda_{k,\varepsilon}$

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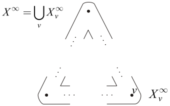

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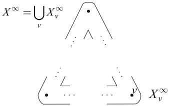

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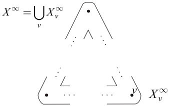

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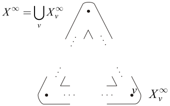

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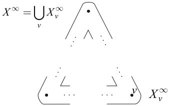

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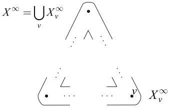

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Let A_v be the $(+1)$-eigenspace of $S_v(\nu)$. Let μ_k be the eigenvalues of the quantum graph G with corner conditions $(A_v)_{v\in V}$. Then the eigenvalues on M_ε are

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For $\nu > 0$ and generic Y_e, X_v one has $A_v = 0$ for all v, i.e. decoupling.

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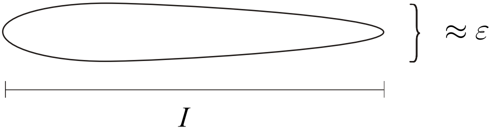

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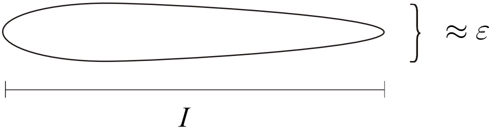

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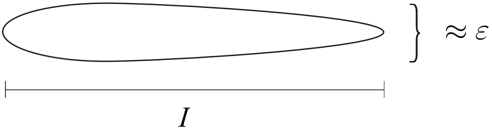

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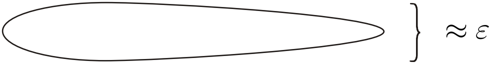
$M \subset \mathbb{R}^2$ convex.



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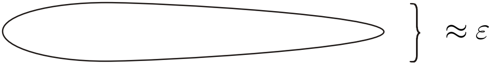

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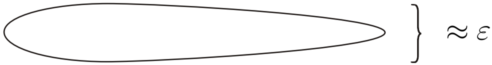

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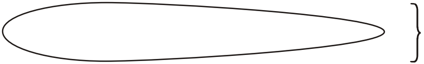

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Let $\operatorname{diam} M = 1$, $\operatorname{inr} M = \varepsilon$.

Theorem (G-Jerison 1996,1998,2007)

- *The location of* $\operatorname{locmax} u_1$ *and of* $u_2^{-1}(0)$ *is determined*
  geometrically *up to an error* C
  by solution of ODE *up to an error* $C\varepsilon$
- *This is optimal in order of magnitude.*

Remark: For optimality need third term $(O(\varepsilon))$ in asymptotics

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