Introduction to the b-calculus: A formal overview

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References

- "Basics of the b-calculus" by D. Grieser, in volume *Approaches to Singular Analysis*, or from Daniel’s webpage.
- *The Atiyah-Patodi-Singer Index Theorem*, R. Melrose, available also on his webpage.
Outline

1 A formal overview
   - The compact setting
   - The noncompact setting
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   - The noncompact setting

2 A simple example
   - From the viewpoint of transforms
   - From the viewpoint of operator kernels
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1. A formal overview
   - The compact setting
   - The noncompact setting

2. A simple example
   - From the viewpoint of transforms
   - From the viewpoint of operator kernels

3. Rigorous definitions
   - The small calculus and the indicial operator
   - The b-trace
   - The full calculus
Why pseudodifferential operators?

ΨDOs are useful for studying differential operators.
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**Theorem**

*If $P$ is an elliptic differential operator on functions (or sections of a bundle) over a compact manifold, then there exists a parametrix $Q$, which is a pseudodifferential operator which inverts $P$ up to compact operators, i.e, $I - PQ = R_r$ and $I - QP = R_l$ are compact.*
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- Structure of spectrum
- Regularity of solutions
- Properties of resolvent
- Properties of associated parabolic, hyperbolic operators, nonlinear operators, etc.
There are two algebras:

(Op) $\Psi = \bigcup_m \Psi^m$ is the graded algebra of pseudodifferential operators:

$$\Psi^m \subset \Psi^{m+1}, \quad \Psi^m \cdot \Psi^n \subset \Psi^{n+m}, \quad \Psi^m + \Psi^n \subset \Psi^m$$

(Symb) $S = \bigcup_m S^m$ is the graded algebra of symbols, with

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And an algebra homomorphism that relates them:

(Hom) $\sigma_m : \Psi^m \rightarrow S^m [m]$
These objects fit together in a short exact sequence:

\[(\text{exact}): \quad 0 \longrightarrow \psi^{m-1} \longrightarrow \psi^m \overset{\sigma_m}{\longrightarrow} S[m] \longrightarrow 0\]
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They relate to differential operators by:

- \((\text{Diff})\) \(Diff^m \subset \Psi^m\)
- \((\text{Ell})\) \(P \in Diff^m\) elliptic implies that \(\sigma_m(P) \in \mathbb{S}^m\) is invertible.
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0 \xrightarrow{} \Psi^{m-1} \xrightarrow{} \Psi^m \xrightarrow{\sigma_m} S^m \xrightarrow{} 0
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They relate to differential operators by:

(Diff) \( \text{Diff}^m \subset \Psi^m \)

(Ell) \( P \in \text{Diff}^m \) elliptic implies that \( \sigma_m(P) \in S^m \) is invertible.

And they relate to Sobolev spaces by:

(Sob) The map \( Q : H^k \longrightarrow H^{k-m} \) is bounded for any \( Q \in \Psi^m \) and \( k \in \mathbb{R} \).
Proof of parametrix theorem in compact setting

By (Ell), if $P \in \text{Diff}^m$ is elliptic, then $\sigma_m(P)$ is invertible, so there is some $\sigma_m(P)^{-1} \in S^{-m}$.
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Then by (Sob) and the Rellich lemma, we have that

$$R_l : H^k \longrightarrow H^{k+1} \longrightarrow H^k$$

is a compact operator, and we are done.
When $P$ is a (uniformly) elliptic differential operator on a noncompact manifold, we can follow exactly the same steps using the calculus of properly supported pseudodifferential operators. Only the very last step fails, because the Rellich lemma does not hold for noncompact manifolds:

$$H^{k+1} \to H^k$$

is not a compact inclusion for a noncompact manifold.
Big Question: Can we get around this problem?
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Not very tidy answer: In many cases (geometries), yes, for instance using tools developed in:

The b-calculus and its variants (Melrose and a cast of thousands)
Other pseudodifferential calculi by Boutet-de-Monvel and Schulze

But no overall solution exists.
The b-calculus ("boundary" calculus) is an extension of the calculus of properly supported pseudodifferential operators which allows the construction of parametrices for uniformly elliptic operators over "b-manifolds".

A b-manifold is a Riemannian manifold with a cylindrical end: on the end, $M - M_0$, the metric is given (approximately) by $g = dt^2 + ds^2$. For $t = 0$, as $t \to \infty$, $M \to M_0$. 

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On the end, \( M - M_0 \), the metric is given (approximately) by \( g = dt^2 + ds_N^2 \).
It is useful to change coordinates and consider $M$ as the interior of a manifold with boundary:

Let $x = e^{-t}$. Then on the end, $M - M_0$, the metric is given (approximately) by

$$g = \frac{dx^2}{x^2} + ds_N^2.$$
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$$g = \frac{dx^2}{x^2} + ds^2_N.$$ 

We can extend the function $x$ to all of $\overline{M}$ to get a \textbf{boundary defining function}. Note that given just $\overline{M}$, there is no canonical choice of $x$. 

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Choose local coordinates \( \{y_1, \ldots, y_n\} \) on \( N \cong \partial M \).

**Definition**

A differential operator of degree \( m \) on \( M \) is called a **b-differential operator** if locally near \( \partial M \), it is of the form

\[
P = \sum_{|\alpha| \leq m} a_\alpha(x, y_1, \ldots, y_n) (x \partial_x)^{\alpha_0} (\partial y_1)^{\alpha_1} \cdots (\partial y_n)^{\alpha_n}
\]

for coefficients \( a_\alpha(x, y_1, \ldots, y_n) \) smooth up to the boundary, \( x = 0 \).
b-Sobolev spaces

**Definition**

If \((M, g)\) is a b-manifold with boundary defining function \(x\), then we define the b-Sobolev spaces in the usual way using the b-metric, \(g\). We also have weighted Sobolev spaces:

\[ x^k H^j_b(M) = \{ f = x^k h \mid h \in H^j_b(M) \}. \]
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If $P$ is a b-differential operator of order $m$, then

$$P : x^k H^j_b(M) \longrightarrow x^k H^{j-m}_b(M)$$

is bounded for all $k, l \in \mathbb{R}$. 
b-Rellich lemma

Lemma

For all $j, k \in \mathbb{R}$, the inclusion

$$x^{k+1} H_{b}^{j+1}(M) \rightarrow x^{k} H_{b}^{j}(M)$$

is compact.

Thus, we want to improve our parametrix so its remainders are bounded maps

$$R_{r/l} : x^{k} H_{b}^{j}(M) \rightarrow x^{k+1} H_{b}^{j+1}(M).$$
We will enlarge the algebra of properly supported operators, denoted $\Psi_b^*$ to various sets of operators, $\Psi_{b,\gamma}^*$.
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For all $k, \gamma, Q \in \Psi_b^m$ is bounded as an operator

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For all $k, \gamma$, $Q \in \Psi^*_b$ is bounded as an operator

$$Q : x^\gamma H^k_b \longrightarrow x^\gamma H^{k-m}_b,$$

$Q \in \Psi^*_b,^\gamma$ is bounded as an operator between $\gamma$ weighted spaces, but not generally.
There are three related graded operator spaces:

\[(\text{Op}) \ \Psi^*_{b,\gamma}(M) = \bigcup_m \Psi^m_{b,\gamma}(M) \text{ and for all } \gamma \in \mathbb{R} \]
\[\Psi^m_b(M) \subset \Psi^m_{b,\gamma}(M).\]
(The parameter is not really in \(\mathbb{R}\), but we can think of it that way to start.)

\[(\text{Symb}) \ S_b = \bigcup_m S^m_b \text{ is the standard symbol space for properly supported operators.}\]

\[(\text{Ind Symb}) \ \Psi^*_{b,\gamma}(\partial M) = \bigcup_m \Psi^m_{b,\gamma}(\partial M) \text{ is a graded calculus of pseudodifferential operators that depends only on } \partial M.\]
There are two homomorphisms relating them:

(Symb) the standard principal symbol

(Ind Symb) the “Indicial operator”

\[ \text{Ind} : \Psi_{b}^{m,\gamma}(M) \longrightarrow \Psi_{b,\Gamma}^{m,\gamma}(\partial M) \]
The formal structure of the b-calculus, II

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\[ \text{Ind} : \psi_{b,\gamma}^m (\mathcal{M}) \rightarrow \psi_{b,\gamma}^m (\partial \mathcal{M}) \]

There are two short exact sequences they fit into:

(Exact) the standard symbol exact sequence
(Ind Exact)

\[ 0 \rightarrow \chi \psi_{b,\gamma}^m (\mathcal{M}) \rightarrow \psi_{b,\gamma}^m (\mathcal{M}) \xrightarrow{\text{Ind}} \psi_{b,\gamma}^m (\partial \mathcal{M}) \rightarrow 0 \]
We’ll define a b-operator as \textit{b-elliptic} if its principal symbol is uniformly invertible, that is, it does not become singular at $\partial M$. 
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(Ell ind) If \( P \in \text{Diff}_b^m \) is b-elliptic then there is a discrete subset \( \mathcal{R}(P) \subset \mathbb{R} \) with \( \text{Ind}(P) \) invertible in \( \Psi_{b,1}^{m,\gamma}(\partial M) \) for any \( \gamma \in \mathbb{R} - \mathcal{R}(P) \).
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Now we can formally construct parametric equivalence for elliptic b-differential operators.
Let $Q_1 \in \Psi_b^{-m}$ be the “small” parametrix for $P$, and let $R_r = I - PQ_1$ following the proof in the compact case.
b-parametrix construction, I

- Let $Q_1 \in \Psi_b^{-m}$ be the “small” parametrix for $P$, and let $R_r = I - PQ_1$ following the proof in the compact case.
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Let $Q_1 \in \Psi^{-m}_b$ be the “small” parametrix for $P$, and let $R_r = I - PQ_1$ following the proof in the compact case.

By (Ell ind), for $\gamma \notin \mathcal{R}(P)$, there is an inverse $\text{Ind}(P)^{-1} \in \Psi^{-m,\gamma}_{b,l}(\partial M)$.

By (Ind exact), there is some $S_\gamma \in \Psi^{-m,\gamma}_b(M)$ with $\text{Ind}(S_\gamma) = \text{Ind}(P)^{-1}$. 
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By (Ind exact), there is some $S_\gamma \in \Psi^{-m,\gamma}_b(M)$ with
$$\text{Ind}(S_\gamma) = \text{Ind}(P)^{-1}.$$

Let $Q_\gamma = S_\gamma R_r$, and let $R'_r = I - P(Q_1 + Q_\gamma)$. Then

$$\text{Ind}(R'_r) = \text{Ind}(I - P(Q_1 + Q_\gamma)) = \text{Ind}(I - PQ_1 - PQ_\gamma)$$

$$= \text{Ind}(R_r - PS_\gamma R_r) = \text{Ind}(R) - \text{Ind}(P)\text{Ind}(P)^{-1}\text{Ind}(R_r) = 0.$$
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$$= \text{Ind}(R_r - PS_\gamma R_r) = \text{Ind}(R) - \text{Ind}(P)\text{Ind}(P)^{-1}\text{Ind}(R_r) = 0.$$ 

Thus by (Ind exact), $R'_r \in \times \Psi_{b}^{-1,\gamma}(M)$. 
So as required:

\[ Q_1 + Q_\gamma : x^\gamma H^k_b(M) \longrightarrow x^\gamma H^{k+m}_b(M) \]

is bounded for all \( k \in \mathbb{R} \), as is

\[ R'_r = xR'' : x^\gamma H^k_b(M) \longrightarrow x^{\gamma+1} H^{k+1}_b(M). \]
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So \( R'_r \) is a compact remainder as a map:

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We can prove analogously that \( R'_r = I - P(Q_1 + Q_\gamma) \) is a compact remainder, so \( Q_1 + Q_\gamma \) is the parametrix we sought.
The parametrix theorem for elliptic b-operators

As in the compact case, it is possible to improve this parametrix construction to arrive at the theorem:

**Theorem**

Let \( P \in \text{Diff}_b^m(M) \) be b-elliptic. Then for each \( \gamma \in \mathbb{R} - \mathcal{R}(P) \), the map

\[
P : x^{\gamma} H^k_b(M) \to x^{\gamma} H^{k-m}_b(M)
\]

is Fredholm with parametrix \( G_\gamma \) and remainder

\[
R_\gamma : x^{\gamma} H^k_b(M) \to x^{\gamma+1} C^\infty(M).
\]

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TOMORROW: A simple example
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The small calculus and the indicial operator
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A formal overview
A simple example
Rigorous definitions
The small calculus and the indicial operator
The b-trace
The full calculus
Introduction to the b-calculus: A formal overview

- A formal overview
- A simple example
- Rigorous definitions
- The small calculus and the indicial operator
- The b-trace
- The full calculus
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