Introduction to the b-calculus: A formal overview

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References

- "Basics of the b-calculus" by D. Grieser, in volume *Approaches to Singular Analysis*, or from Daniel's webpage.
- *The Atiyah-Patodi-Singer Index Theorem*, R. Melrose, available also on his webpage.

Outline

1. A formal overview
   - The compact setting
   - The noncompact setting

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2. A simple example
   - From the viewpoint of transforms
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2. A simple example
   - From the viewpoint of transforms
   - From the viewpoint of operator kernels
3. Rigorous definitions
   - The small calculus and the indicial operator
   - The b-trace
   - The full calculus

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If P is an elliptic differential operator on functions (or sections of a bundle) over a compact manifold, then there exists a parametrix Q, which is a pseudodifferential operator which inverts P up to compact operators, i.e, $I - PQ = R_r$ and $I - QP = R_l$ are compact.

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- Structure of spectrum
- Regularity of solutions
- Properties of resolvent
- Properties of associated parabolic, hyperbolic operators, nonlinear operators, etc.

Formal properties of the ΨDO algebra over a compact manifold, I

There are two algebras:

(Op) $\Psi = \cup_m \Psi^m$ is the graded algebra of pseudodifferential operators:

$$\Psi^m \subset \Psi^{m+1}, \qquad \Psi^m \cdot \Psi^n \subset \Psi^{n+m}, \qquad \Psi^m + \Psi^m \subset \Psi^m$$

(Symb) $S = \cup_m S^m$ is the graded algebra of symbols, with

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And an algebra homomorphism that relates them:

(Hom) $\sigma_m : \Psi^m \longrightarrow S^{[m]}$

Formal properties of the ΨDO algebra over a compact manifold, II

These objects fit together in a short exact sequence:

(exact):

$$0 \longrightarrow \Psi^{m-1} \longrightarrow \Psi^m \xrightarrow{\sigma_m} S^{[m]} \longrightarrow 0$$

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(Diff) $\mathit{Diff}^m \subset \Psi^m$
(Ell) $P \in \mathit{Diff}^m$ elliptic implies that $\sigma_m(P) \in S^{[m]}$ is invertible.

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(Diff) $\mathit{Diff}^m \subset \Psi^m$
(Ell) $P \in \mathit{Diff}^m$ elliptic implies that $\sigma_m(P) \in S^{[m]}$ is invertible.

And they relate to Sobolev spaces by:

(Sob) The map $Q : H^k \longrightarrow H^{k-m}$ is bounded for any $Q \in \Psi^m$ and $k \in \mathbb{R}$.

Proof of parametrix theorem in compact setting

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Then by (Sob) and the Rellich lemma, we have that

$$R_l : H^k \longrightarrow H^{k+1} \longrightarrow H^k$$

is a compact operator, and we are done.

Trouble in the noncompact setting

When P is a (uniformly) elliptic differential operator on a noncompact manifold, we can follow exactly the same steps using the calculus of properly supported pseudodifferential operators.

Only the very last step fails, because the Rellich lemma does not hold for noncompact manifolds:

$$H^{k+1} \longrightarrow H^k$$

is not a compact inclusion for a noncompact manifold.

Where now?

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Not very tidy answer: In many cases (geometries), yes, for instance using tools developed in:

The b-calculus and its variants (Melrose and a cast of thousands)
Other pseudodifferential calculi by Boutet-de-Monvel and Schulze

But no overall solution exists.

b-manifolds, I

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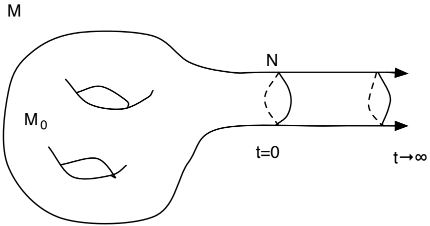
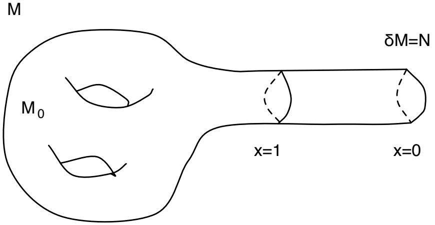
A b-manifold is a Riemannian manifold with a cylindrical end:



On the end, $M - M_0$, the metric is given (approximately) by $g = dt^2 + ds_N^2$.

b-manifolds, II

It is useful to change coordinates and consider M as the interior of a manifold with boundary:



Let $x = e^{-t}$. Then on the end, $M - M_0$, the metric is given (approximately) by

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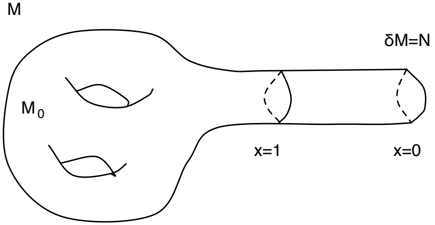

Let $x = e^{-t}$. Then on the end, $M - M_0$, the metric is given (approximately) by

$$g = \frac{dx^2}{x^2} + ds_N^2.$$

We can extend the function x to all of $\overline{M}$ to get a boundary defining function. Note that given just $\overline{M}$, there is no canonical choice of x.

b-differential operators

Choose local coordinates $\{y_1, \ldots, y_n\}$ on $N \cong \partial M$.

Definition

A differential operator of degree m on M is called a b-differential operator if locally near ∂M, it is of the form

$$P = \sum_{|\alpha| \leq m} a_\alpha(x, y_1, \ldots, y_n)(x\partial_x)^{\alpha_0}(\partial_{y_1})^{\alpha_1} \cdots (\partial_{y_n})^{\alpha_n}$$

for coefficients $a_\alpha(x, y_1, \ldots, y_n)$ smooth up to the boundary, $x = 0$.

b-Sobolev spaces

Definition

If (M, g) is a b-manifold with boundary defining function x, then we define the b-Sobolev spaces in the usual way using the b-metric, g. We also have weighted Sobolev spaces:

$$x^k H_b^j(M) = \{f = x^k h \mid h \in H_b^j(M)\}.$$

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$$x^k H_b^j(M) = \{f = x^k h \mid h \in H_b^j(M)\}.$$

If P is a b-differential operator of order m, then

$$P : x^k H_b^j(M) \longrightarrow x^k H_b^{j-m}(M)$$

is bounded for all $k, l \in \mathbb{R}$.

b-Rellich lemma

Lemma

For all $j, k \in \mathbb{R}$, the inclusion

$$x^{k+1} H_b^{j+1}(M) \longrightarrow x^k H_b^j(M)$$

is compact.

Thus, we want to improve our parametrix so its remainders are bounded maps

$$R_{r/l} : x^k H_b^j(M) \longrightarrow x^{k+1} H_b^{j+1}(M).$$

Preliminary notes on the b-calculus

- We will enlarge the algebra of properly supported operators, denoted Ψ_b^* to various sets of operators, $\Psi_b^{*,\gamma}$.

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- $Q \in \Psi_b^{*,\gamma}$ is bounded as an operator between γ weighted spaces, but not generally.

The formal structure of the b-calculus,I

There are three related graded operator spaces:

(Op) $\Psi_b^{*,\gamma}(M) = \cup_m \Psi_b^{m,\gamma}(M)$ and for all $\gamma \in \mathbb{R}$
$\Psi_b^m(M) \subset \Psi_b^{m,\gamma}(M)$.
(The parameter is not really in $\mathbb{R}$, but we can think of it that way to start.)

(Symb) $S_b = \cup_m S_b^m$ is the standard symbol space for properly supported operators.

(Ind Symb) $\Psi_{b,I}^{*,\gamma}(\partial M) = \cup_m \Psi_{b,I}^{m,\gamma}(\partial M)$ is a graded calculus of pseudodifferential operators that depends only on ∂M.

The formal structure of the b-calculus,II

There are two homomorphisms relating them:

(Symb) the standard principal symbol

(Ind Symb) the "Indicial operator"

$$Ind : \Psi_b^{m,\gamma}(M) \longrightarrow \Psi_{b,I}^{m,\gamma}(\partial M)$$

The formal structure of the b-calculus,II

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(Symb) the standard principal symbol

(Ind Symb) the "Indicial operator"

$$Ind : \Psi_b^{m,\gamma}(M) \longrightarrow \Psi_{b,I}^{m,\gamma}(\partial M)$$

There are two short exact sequences they fit into:

(Exact) the standard symbol exact sequence

(Ind Exact)

$$0 \to x\Psi_b^{m,\gamma}(M) \to \Psi_b^{m,\gamma}(M) \xrightarrow{Ind} \Psi_{b,I}^{m,\gamma}(\partial M) \to 0$$

The formal structure of the b-calculus, III

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The formal structure of the b-calculus, III

We'll define a b-operator as b-elliptic if its principal symbol is uniformly invertible, that is, it does not become singular at ∂M. Then the new spaces relate to b-elliptic operators through:

(Ell ind) If $P \in Diff_b^m$ is b-elliptic then there is a discrete subset $\mathcal{R}(P) \subset \mathbb{R}$ with $Ind(P)$ invertible in $\Psi_{b,I}^{m,\gamma}(\partial M)$ for any $\gamma \in \mathbb{R} - \mathcal{R}(P)$.

The formal structure of the b-calculus, III

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Now we can formally construct parametrices for elliptic b-differential operators.

b-parametrix construction, I

- Let $Q_1 \in \Psi_b^{-m}$ be the "small" parametrix for P, and let $R_r = I - PQ_1$ following the proof in the compact case.

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- By (Ind exact), there is some $S_\gamma \in \Psi_b^{-m,\gamma}(M)$ with $Ind(S_\gamma) = Ind(P)^{-1}$.
- Let $Q_\gamma = S_\gamma R_r$, and let $R_r' = I - P(Q_1 + Q_\gamma)$. Then

$$Ind(R_r') = Ind(I - P(Q_1 + Q_\gamma)) = Ind(I - PQ_1 - PQ_\gamma)$$

$$= Ind(R_r - PS_\gamma R_r) = Ind(R) - Ind(P)Ind(P)^{-1}Ind(R_r) = 0.$$

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- Let $Q_\gamma = S_\gamma R_r$, and let $R_r' = I - P(Q_1 + Q_\gamma)$. Then

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$$= Ind(R_r - PS_\gamma R_r) = Ind(R) - Ind(P)Ind(P)^{-1}Ind(R_r) = 0.$$

- Thus by (Ind exact), $R_r' \in x\Psi_b^{-1,\gamma}(M)$.

b-parametrix construction, II

So as required:

$$Q_1 + Q_\gamma : x^\gamma H_b^k(M) \longrightarrow x^\gamma H_b^{k+m}(M)$$

is bounded for all $k \in \mathbb{R}$, as is

$$R_r' = xR'' : x^\gamma H_b^k(M) \to x^{\gamma+1} H_b^{k+1}(M).$$

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We can prove analogously that $R_l' = I - P(Q_1 + Q_\gamma)$ is a compact remainder, so $Q_1 + Q_\gamma$ is the parametrix we sought.

The parametrix theorem for elliptic b-operators

As in the compact case, it is possible to improve this parametrix construction to arrive at the theorem:

Theorem

Let $P \in \mathrm{Diff}_b^m(M)$ be b-elliptic. Then for each $\gamma \in \mathbb{R} - \mathcal{R}(P)$, the map

$$P : x^\gamma H_b^k(M) \to x^\gamma H_b^{k-m}(M)$$

is Fredholm with parametrix G_γ and remainder

$$R_\gamma : x^\gamma H_b^k(M) \to x^{\gamma+1} C^\infty(M).$$

Coming attractions

TOMORROW: A simple example