Introduction to the b-calculus: A simple example

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Outline

Yesterday A formal overview

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Today A simple example

- The question
- Solution from the viewpoint of transforms
- Some generalizations
- Solution from the viewpoint of operator kernels

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Tomorrow Rigorous definitions

The question

We want to understand parametrices of uniformly elliptic differential operators on noncompact manifolds (elliptic b-differential operators on b-manifolds, in particular), so today we will consider the very simplest such operator:

$P = \frac{d}{ds}$ on $C_0^\infty(\mathbb{R})$, extended to its natural domain

$$H^1(\mathbb{R}) \subset L^2(\mathbb{R}).$$

Question: Can we find a parametrix for P, i.e., can we invert $P : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ up to a compact operator?

Naïve answer

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and

$$Q(Pg)(s) = \int_0^s g'(r)\,dr = g(s) - g(0).$$

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So formally, Q is a parametrix, since

$$I - PQ = 0.$$

and

$$I - QP = \Pi_{\text{constant functions}}$$

Problem

The problem with this Q is that $g \in C_0^\infty(\mathbb{R})$ does not imply that $Qg \in L^2(\mathbb{R})$. So Q is not a parametrix for $P : H^1(\mathbb{R}) \to L^2(\mathbb{R})$.

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New Question: Can we find spaces D and R so that $P : D \to R$ is Fredholm between these spaces?

The eventual result:

Definition

Define the weighted Sobolev spaces

$$\phi_{-\alpha,\alpha}H^k(\mathbb{R}) = \{f = e^{-\alpha r}g | g \in H^k(\mathbb{R})\}.$$

Note: if we compactify $\mathbb{R}$, these are b-Sobolev spaces with weight $x^{\pm\alpha}$ at the boundary components at $\pm\infty$.

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Theorem

The operator $P : \phi_{-\alpha,\alpha}H^1(\mathbb{R}) \longrightarrow \phi_{-\alpha,\alpha}L^2(\mathbb{R})$ is Fredholm with a parametrix Q_α if and only if $\alpha \neq 0$.

Proof

Let $f(r) \in C_0^\infty(\mathbb{R})$ =smooth compactly supported functions on $\mathbb{R}$. Then using Fourier transforms we get:

$$f(r) = (f(r')^\wedge)^\vee = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\eta(r-r')} f(r')\, dr' d\eta.$$

Further, we know it is possible to extend the Fourier and inverse Fourier transforms to functions $f(r) \in L^2(\mathbb{R})$.

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Now for $f(r) \in C_0^\infty(\mathbb{R})$, consider

$$f(r) = e^{-\alpha r}((e^{\alpha r'} f(r'))^\wedge)^\vee = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(\eta + i\alpha)(r-r')} f(r')\, dr'\, d\eta.$$

This can be extended to functions $f(r')$ such that $e^{\alpha r'} f(r') \in L^2(\mathbb{R})$, that is, to $f \in \phi_{-\alpha,\alpha} L^2(\mathbb{R})$.

The Shifted Fourier Transform

We can rearrange this, and use it to define a new transform:

Definition

Let $\alpha \in \mathbb{R}$. The shifted Fourier transform, M_α is defined on $f \in C_0^\infty$ by:

$$M_\alpha f(\eta) = \int_{\mathbb{R}} e^{-i(\eta+i\alpha)r'} f(r')\, dr'.$$

The inverse transform, M_α^{-1} is defined on $g \in C_0^\infty$ by:

$$M_\alpha^{-1} g(r) = \int_{\mathbb{R}} e^{i(\eta+i\alpha)r} g(\eta)\, d\eta.$$

These transforms extend to maps between $\phi_{-\alpha,\alpha} L^2(\mathbb{R})$ and $L^2(\mathbb{R})$.

Properties of the shifted Fourier transform

The shifted Fourier transform inherits nice properties from the standard Fourier transform:

- $M_\alpha(f+g) = M_\alpha(f) + M_\alpha(g)$
- $M_\alpha(cf) = cM_\alpha(f)$

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Note: If we make the change of coordinates $x = e^{-r}$ the shifted Fourier transform becomes the Mellin transform. We will use these coordinates, and the term "Mellin transform" tomorrow.

Building the parametrix, Q_α

We want to solve $u' = f$ for $f \in C_0^\infty$. Take M_α of both sides:

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Since $\eta + i\alpha \neq 0$ for any real $\alpha \neq 0$, we can divide and inverse transform to get:

$$Q_\alpha f(r) := u(r) = M_\alpha^{-1}(\frac{1}{\eta + i\alpha} M_\alpha f(\eta)).$$

End of proof

Further, from the Fourier transform properties we can show:

$$||Q_\alpha f||_{\phi_{-\alpha,\alpha}H^1} \leq |\alpha|^{-1}||f||_{\phi_{-\alpha,\alpha}L^2}$$

So Q_α extends to a bounded map $Q_\alpha : \phi_{-\alpha,\alpha}H^1 \to \phi_{-\alpha,\alpha}L^2$.

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So Q_α extends to a bounded map $Q_\alpha : \phi_{-\alpha,\alpha}H^1 \to \phi_{-\alpha,\alpha}L^2$.

Finally, we notice that constant functions are not in $\phi_{-\alpha,\alpha}L^2$, so P has no kernel, and is invertible on these spaces with inverse, Q_α.

We know what the inverse is for P on these spaces–integration starting at ∞ if $\alpha > 0$ and at $-\infty$ if $\alpha < 0$.

Generalizations, I

We can generalize this result slightly to give us a better intuition about b-differential operators and their parametrices in general.

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Different weights: It is possible to weight the two ends differently.

Let $\phi_{\alpha,\beta}$ behave like $e^{\alpha r}$ near $-\infty$ and like $e^{-\beta r}$ near ∞. Then if $\alpha \neq 0$ and $\beta \neq 0$, there is a parametrix

(though not generally an inverse) $Q_{\alpha,\beta}$ for $P : \phi_{\alpha,\beta} H^1(\mathbb{R}) \to \phi_{\alpha,\beta} L^2(\mathbb{R})$.

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(though not generally an inverse) $Q_{\alpha,\beta}$ for $P : \phi_{\alpha,\beta} H^1(\mathbb{R}) \to \phi_{\alpha,\beta} L^2(\mathbb{R})$.

Add a constant: If $P_c = \frac{d}{ds} + c$, then we have $e^{cs} P_c e^{-cs} = \frac{d}{ds}$, but multiplying by $e^{\pm cs}$ shifts weights, so we have:

$$P_c : \phi_{\alpha,\beta} H^1(\mathbb{R}) \to \phi_{\alpha,\beta} L^2(\mathbb{R})$$

has a parametrix iff $\alpha \neq -c$ and $\beta \neq c$.

Generalizations, II

Vector functions: Assume that A, B are $k \times k$ matrices over $\mathbb{C}$, and let $P_{A,B} = A\frac{d}{ds} + B$. Then $P_{A,B}$ is elliptic if A is invertible. Assume further that $A^{-1}B$ is diagonalizable.

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Then we can solve

$$P_{A,B}\mathbf{u} = \mathbf{v}$$

by rewriting it as

$$(\frac{d}{ds} + A^{-1}B)\mathbf{u} = A^{-1}\mathbf{v}$$

Generalizations, III

Then diagonalize to get a vector of equations of the form

$$(\frac{d}{ds} + \lambda_j)u_j = (A^{-1}\mathbf{v})_j.$$

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These are equations like the last generalization, so there is a bounded solution operator for this vector of equations

$$Q_{\alpha,\beta} : \phi_{\alpha,\beta}H^1(\mathbb{R}, \mathbb{C}^k) \to \phi_{\alpha,\beta}L^2(\mathbb{R}, \mathbb{C}^k)$$

so long as $-\alpha \neq Re(\lambda_j) \neq \beta$ for any eigenvalue λ_j of $A^{-1}B$.

Final generalization

Finally, we can allow the coefficients to vary in our operator.

Theorem

Let

$$P = A(s)\frac{d}{ds} + B(s) : \phi_{\alpha,\beta} H^1(\mathbb{R}, \mathbb{C}^k) \to \phi_{\alpha,\beta} L^2(\mathbb{R}, \mathbb{C}^k)$$

Assume

- *$A(s)$ converges exponentially at $\pm\infty$ to $A_\pm$ and $B(s)$ converges exponentially at $\pm\infty$ to $B_\pm$*
- *$A(s)$ is invertible for all s and $A_\pm$ are also invertible*
- *$A(s)$ and $B(s)$ are smooth on $\overline{\mathbb{R}}$.*

Then P is Fredholm iff α is not an eigenvalue of $-(A_-)^{-1}B_-$ and β is not an eigenvalue of $(A_+)^{-1}B_+$.

Morals

Three important ideas from yesterday's formal treatment are thus revealed also in the case of first order elliptic b-operators over $\mathbb{R}$:

- The Fredholm properties of the operators depend on weights at infinity in the domain and range.

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Three important ideas from yesterday's formal treatment are thus revealed also in the case of first order elliptic b-operators over $\mathbb{R}$:

- The Fredholm properties of the operators depend on weights at infinity in the domain and range.
- Elliptic b-operators are Fredholm for weights outside a discrete "bad" set.
- The "bad" weights come from the spectral theory of limiting operators on a cross section at infinity.

Standard ΨDO kernels

Recall that the basis of the standard ΨDO algebra is the Schwartz kernel theorem, which states that any linear operator acting on $C_0^\infty(M)$ can be realized as integration against a distribution on M^2, called its operator kernel, so

$$Pf(x) = \int_M K_P(x,y)f(y)dy.$$

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Pseudodifferential operators are then defined to be operators whose integral kernels have *singular support* along the diagonal, although their smooth support may be anything. Again, their degree as operators is related to the singularities of their kernels along Δ.

The operator kernel for Q_c, I

The b-calculus, like the standard ΨDO calculus, is described in terms of integral kernels. Recall from the first generalization how we constructed the parametrix Q_c for $P_c = \frac{d}{dr} + c$:

$$Q_c f(r) = e^{cr} Q(e^{-cr'} f(r'))(r)$$

$$= e^{cr} \int_{-\infty}^{r} e^{-cr'} f(r')\, dr'.$$

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In order to study the behaviour of K_{Q_c} we want to bring $\pm\infty$ in and compactify $\mathbb{R}$ as $[0, 1]$, so we let

$$x = \frac{e^r}{1 + e^r}, \qquad x' = \frac{e^{r'}}{1 + e^{r'}}.$$

The operator kernel for Q_c, II

In these new coordinates, $P_c = x(1-x)\frac{d}{dx} + c$ is clearly an elliptic b-differential operator and

$$Q_c(f)(x) = \int_0^x \left(\frac{x}{1-x}\right)^c \left(\frac{x'}{1-x'}\right)^{-c} f(x') \frac{dx'}{x'(1-x')}$$

$$= \int_0^1 K_{Q_c}(x,x') f(x') \frac{dx'}{x'(1-x')},$$

which is an integral with respect to the b-volume form against the integral kernel:

$$K_{Q_c} = \frac{x^c}{(1-x)^c} \frac{(1-x')^c}{(x')^c} (1 - H(x'-x)).$$

Singularities of K_{Q_c}

In the standard pseudodifferential calculus:

- operator kernels are smooth except along the diagonal in $M \times M$,
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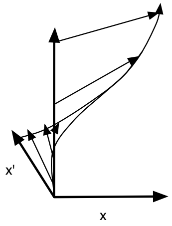
- conormal singularity along the diagonal
- power-type singularities at boundaries $x = 0, 1$ and $x' = 0, 1$
- worse singularities at $(0, 0)$ and $(1, 1)$.

Motivation for blowing up

At $(0,0)$, we have $x^c/(x')^c$, which has no limit:



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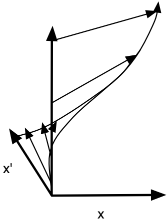
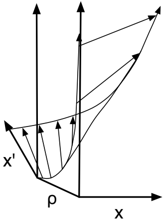


So we blow up the corner using $\rho = x/x'$ to get a better singularity at the resulting submanifold:



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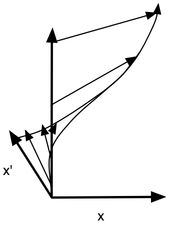

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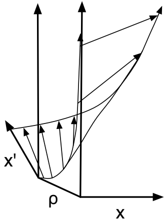
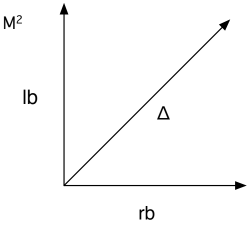


On blown up space, $K_{Q_c} = \rho^c(1 - H(\rho - 1)) \left(\frac{1-x'}{1-\rho x'}\right)^c$.

Moral: The b-double space

In the standard ΨDO calculus:

- Operators are given by kernels on M^2
- Kernels have conormal singularities on Δ:



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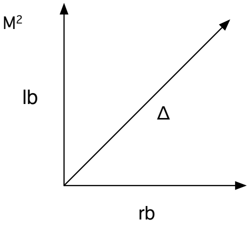
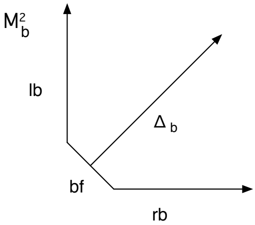


In the b-calculus:

- Operators are given by kernels on M_b^2
- Kernels have conormal singularities on Δ_b
- Kernels have power-type singularities at *rb*, *lb* and *bf*:



Next time

Next time I will build on the intuition from today's examples to fill out Monday's formal b-calculus structure more rigorously.