Introduction to the b-calculus: Rigorous definitions

Eugénie Hunsicker

Loughborough University

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Outline

Monday A formal overview

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Yesterday A simple example

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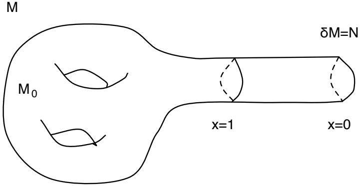
Today Rigorous definitions

- The small calculus and the indicial operator
- The b-trace
- The full calculus

Exact b-metrics

Recall a b-manifold is a compact manifold M with boundary $\partial M = N$ with a boundary defining function x: $x|_{\partial M} = 0$, $dx|_{\partial M} \neq 0$.



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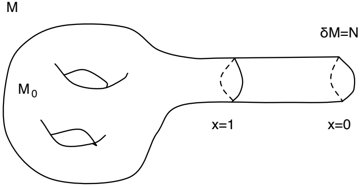

Definition

An exact b-metric on M is a metric g which locally near ∂M has the form:

$$g = (1 + \mathcal{O}(x^2))\frac{dx^2}{x^2} + \sum_i \mathcal{O}(x^2)\frac{dx}{x} \otimes dy_i + g_N(x)$$

where the y_i are local coordinates on N and $g_N(x)$ is smooth up to $x = 0$.

b-Elliptic operators

Recall that a b-differential operator on M of order m is a differential operator on M which locally near ∂M, is of the form

$$P = \sum_{|\alpha| \leq m} a_\alpha(x, y_1, \ldots, y_n)(x\partial_x)^{\alpha_0}(\partial_{y_1})^{\alpha_1} \cdots (\partial_{y_n})^{\alpha_n}$$

for coefficients $a_\alpha(x, y_1, \ldots, y_n)$ smooth up to the boundary, $x = 0$.

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Definition

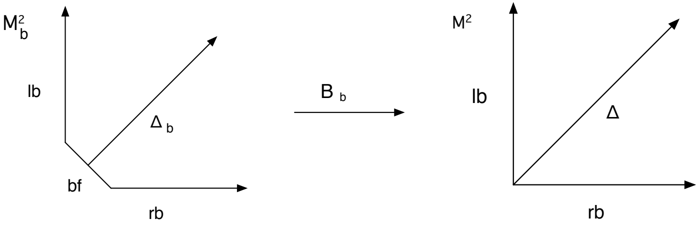
The b-principal symbol of P is then the polynomial

$$^b\sigma_m(P) = \sum_{|\alpha|=m} a_\alpha(x, y_1, \ldots, y_n)\lambda^{\alpha_0}\eta_1^{\alpha_1} \cdots \eta_n^{\alpha_n}.$$

We say that P is b-elliptic if $^b\sigma_m(P)$ is invertible for $(\lambda, \eta_1, \ldots, \eta_n) \neq 0$.

Integral kernels on the b-double space

We discussed last time that we will describe operators in the b-calculus in terms of integral kernels living on the blown up double space, M_b^2:



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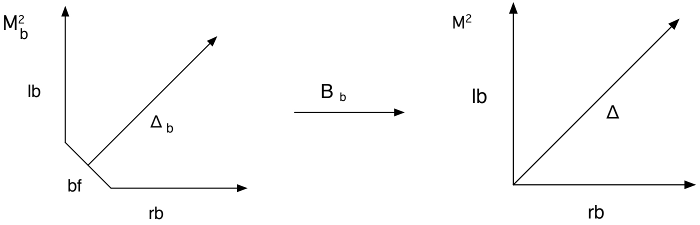

Question: How does such a kernel act on functions on M?

Action of standard kernels, I

Recall that a standard pseudodifferential operator T on a compact manifold X is given by an integral kernel, which is a distribution on X^2. Then T acts on functions by

$$Tf(t) = \int_X K_T(t,t')f(t')dt'.$$

We can think of this action in terms of pull backs and pushforwards. There are two natural maps

$$\pi_{r/l} : X^2 \to X,$$

given by projection onto the left and right coordinates of X^2.

Action of standard kernels, II

To find $Tf(t)$ we first *pull back* the function $f(t')$ from the left copy of X to X^2. The function $\pi_l^*(f)$ is really just the same function $f(t')$, but now we think of it as a function of both t and t' that just happens not to depend on t, i.e., $\pi_l^*(f)(t,t') = f(t')$.

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Finally we *push forward* to get the result:

$$T(f)(t) = (\pi_r)_*(K_T(t,t')\pi_l^*(f)) = (\pi_r)_*(K_T(t,t')f(t')).$$

Pushing forward just means integrating over the fibres of the map π_r, which here is just integrating in t':

$$= \int_X K_T(t,t')f(t')\,dt'.$$

Action of b-kernels

Now consider the b-calculus, where an operator Q is given by a distribution K_Q on M_b^2.

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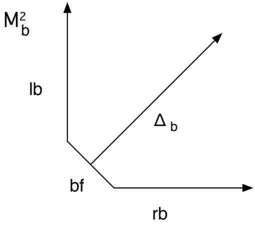
- Multiply by K_Q
- Push forward the product (integrate along fibres of) $M_b^2 \xrightarrow{B_b} M^2 \xrightarrow{\pi_r} M$.

Singularities of integral kernels

Thus singularities of K_Q at various submanifolds in M_b^2 determine various mapping properties for Q:



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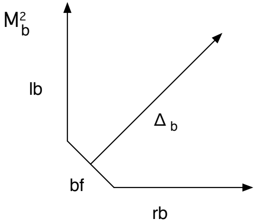

- As in the compact case, regularity of K_Q along the diagonal, Δ_b determines the degree of Q, i.e., the difference in regularity between natural domain and range.

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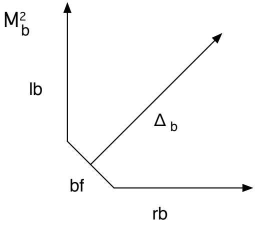

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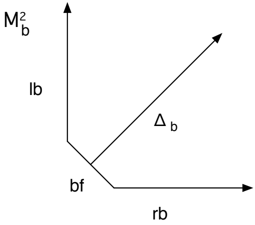

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- Asymptotics at *rb* and *bf* determine the weight of the natural domain of Q.
- Asymptotics at *lb* and *bf* determine the range of Q, absolutely and relative to the domain, respectively.

The small calculus, I

The small b-calculus can be thought of as the calculus of properly supported ΨDOs on M.

Recall properly supported ΨDOs take compactly supported functions to compactly supported functions, although they may enlarge support. To do this, K_Q must be supported away from lb and rb.

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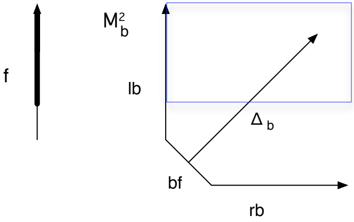

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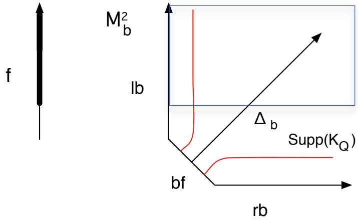

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- Pull f back to M_b^2
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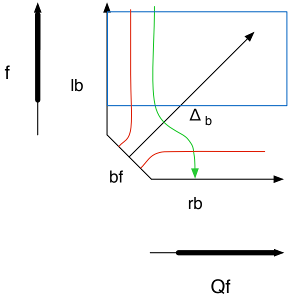
- Push product forward to M



The small calculus, II

Definition

The small b-calculus, $\Psi_b^*(M)$ is defined to be those operators Q whose kernels K_Q are

- conormal along the lifted diagonal, Δ_b,
- vanish to all orders at *lb* and *rb*, and
- are smooth up to *bf* in the normal directions.

Note that any b-differential operator has a kernel supported on Δ_b where it has a conormal singularity, and where the coefficients are smooth in the directions normal to bf. Thus any b-differential operator is in the small b-calculus.

Kernel of the small remainder

Let Q_1 be the small parametrix for P. Then the small remainder, $R_I = I - QP$ is in the small calculus, and its kernel K_{R_I} is smooth everywhere, and need not vanish at bf.

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For an operator R to be compact, its kernel K_R must vanish at bf.

Thus we define a new "symbol", Ind that is an obstruction to vanishing at bf. Then for an operator R to be compact, we will need both that K_R is smooth enough along Δ_b and that $Ind(R) = 0$.

The indicial operator

Definition

The indicial operator, $Ind(P)$ for an element $P \in \Psi_b^m(M)$ is an operator over $\partial M \times \mathbb{R}$. It is obtained by restricting K_P to $bf \cong \mathbb{R} \times \partial M \times \partial M$ and treating this as a convolution kernel in the $\mathbb{R}$ direction and a standard kernel in the ∂M direction.

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If P is a b-differential operator,

$$P = \sum_{|\alpha| \leq m} a_\alpha(x, y_1, \ldots, y_n)(x\partial_x)^{\alpha_0}(\partial_{y_1})^{\alpha_1} \cdots (\partial_{y_n})^{\alpha_n}$$

then

$$Ind(P) = \sum_{|\alpha| \leq m} a_\alpha(0, y_1, \ldots, y_n)(\partial_t)^{\alpha_0}(\partial_{y_1})^{\alpha_1} \cdots (\partial_{y_n})^{\alpha_n}$$

The indicial exact sequence

Suppose that T is an element of the small calculus. Since all kernels must be smooth up to bf in the normal directions, the only way for $Ind(T)$ to vanish is if it actually vanishes to order 1 at $x = 0$, ie, if $T = xS$ for some b-operator S.

Thus we have an exact sequence:

$$0 \to x\Psi_b^m(M) \to \Psi_b^m(M) \stackrel{Ind}{\to} \Psi_{b,I}^m(\partial M) \to 0.$$

The trace functional

Before continuing with the b-calculus construction, we will consider the b-trace, which Pat MacDonald will use.

Recall for compact M:

- any smoothing operator A is trace class
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On a b-manifold, we can break up this integral:

$$Tr(A) = \int_{M_0} K_A(s,s)\, dvol_M + \int_0^1 \int_{\partial M} K_A(x,y,x,y)\, dvol_{\partial M} \frac{dx}{x}.$$

Thus this integral also only converges if K_A vanishes at bf, that is, if $Ind(A) = 0$.

The b-trace

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Definition

If x is a boundary defining function for M, let ν be the trivialization of the normal bundle to $\partial M \subset M$ that makes $dx \cdot \nu = 1$ on ∂M. Then define

$$^{\nu}\!\int_M \phi = \lim_{\epsilon \to 0^+} \left(\int_{x > \epsilon} \phi \, dvol_M + \log(\epsilon) \int_{\partial M} \phi|_{\partial M} \, dvol_{\partial M} \right).$$

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Example: If ϕ is a polynomial near ∂M, e.g. $\phi = (x^2 + x + 2)$, this kills the singular effect of the 0^{th} order term. In particular, on $M \times I$, if ϕ is constant in x, then its regularized integral vanishes.

Non-cannonical

This regularized integral is not cannonical. If we change our boundary defining function x to $x' = x/a$ (or equivalently our trivialization ν to $a\nu$), we get:

$$
{}^{a\nu}\!\int_M \phi = \lim_{\epsilon/a \to 0^+} \left(\int_{x' > \epsilon/a} \phi + \log(\epsilon/a) \int_{\partial M} \phi|_{\partial M} \right).
$$

$$
= \lim_{\epsilon \to 0^+} \left(\int_{x > \epsilon} \phi + \log(\epsilon) \int_{\partial M} \phi|_{\partial M} - \log(a) \int_{\partial M} \phi|_{\partial M} \right).
$$

$$
= -\log(a) \int_{\partial M} \phi|_{\partial M} + {}^{\nu}\!\int_M \phi.
$$

b-trace

Nevertheless, we use it to define the (also non-cannonical) b-trace functional:

Definition

If ν is a trivialization of the normal bundle to $\partial M \subset M$ and $A \in \Psi_b^{-\infty}$, then

$$b{-}Tr_\nu(A) = {}^{\nu}\!\!\int_M K_A|_{\Delta_b} dvol_M.$$

b-trace properties

- If $Ind(A) = 0$, so $A \in x\Psi_b^{-\infty}$then $b{-}Tr_\nu(A) = Tr(A)$ and this is independent of ν.
- If K_A is independent of x, so $A \in \Psi_{b,I}^{-\infty}$ then the b-integral vanishes, so $b{-}Tr(A) = 0$.

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- Thus the following diagram is commutative:

$$\begin{array}{ccccc} x\Psi_b^{-\infty}(M) & \longrightarrow & \Psi_b^{-\infty}(M) & \xrightarrow{Ind} & \Psi_{b,I}^{-\infty}(\partial M) \\ \downarrow{\scriptstyle Tr} & & \downarrow{\scriptstyle b\text{–}Tr} & & \downarrow{\scriptstyle b\text{–}Tr} \\ \mathbb{C} & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \end{array}$$

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- But warning: the b-trace is not really a trace, in that $b{-}Tr_\nu([A, B]) \neq 0$.

The Mellin transform of $Ind(P)$

Recall from yesterday that we inverted the operator $\frac{d}{dr}$ for weighted Sobolev spaces over $\mathbb{R}$ using the Mellin transform. More generally, we can take the Mellin transform of $Ind(P)$ for any elliptic b-differential operator.

For

$$P = \sum_{|\alpha| \leq m} a_\alpha(x, y_1, \ldots, y_n)(x\partial_x)^{\alpha_0}(\partial_{y_1})^{\alpha_1} \cdots (\partial_{y_n})^{\alpha_n}$$

then

$$\widehat{Ind(P)} = \sum_{|\alpha| \leq m} a_\alpha(0, y_1, \ldots, y_n)(\eta + i\gamma)^{\alpha_0}(\partial_{y_1})^{\alpha_1} \cdots (\partial_{y_n})^{\alpha_n}.$$

is a family of differential operators on ∂M parametrized by $\mathbb{C}$.

Inverting $Ind(P)$, I

Recall also from our example of $P = \frac{d}{dr}$ that $Ind(P)$ could only be inverted for certain weights. This generalizes to the following theorem:

Theorem

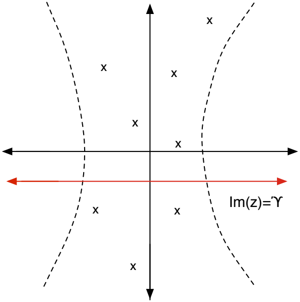
If $P \in Diff_b^m(M)$ is an elliptic b-differential operator, then there is a discrete subset in $\mathbb{C}$:

$$spec_b(P) = \{z \in \mathbb{C} | \widehat{Ind(P)}(z) \text{ is not invertible on } C^\infty(\partial M)\}$$

Further, the set of γ such that $z = \gamma + i\eta \in spec_b(Ind(P))$ for some η is discrete in $\mathbb{R}$. The family of inverses $\widehat{Ind(P)}^{-1}$ is meromorphic in $\mathbb{C}$ with finite rank, finite order poles with finite rank smoothing operators as residues.

Inverting $Ind(P)$, II

The set $spec_b(Ind(P))$ looks like:



So you can invert $Ind(P)$ by taking the inverse Mellin transform along any line with fixed imaginary part γ that does not contain any poles of $\widehat{Ind(P)}(z)$.

Inverting $Ind(P)$, III

The kernel of the resulting inverse, $Ind(P)_\gamma^{-1}$ will have an expansion at *lb* determined by the poles above the line $Im(z) = \gamma$ and at *rb* determined by the poles below this line.

Then there will be an operator Q_γ over M with $Ind(Q_\gamma) = Ind(P)_\gamma^{-1} Ind(R_L)$ and it will be in the space:

Definition

$\Psi_b^{-\infty,\gamma}(M) :=$ operators with smooth kernels on M_b^2 up to *bf* that have expansions in products of powers and powers of logs of the boundary defining functions at *lb* and *rb*, vanishing to at least order γ at *lb* and at least order $-\gamma$ at *rb*.

These correspond to bounded operators:

$$x^\gamma H^k(M) \to x^\gamma C^\infty(M).$$

The full parametrix for P

Thus the parametrix, $Q_1 + Q_\gamma$ for P lives in

$$\Psi_b^{-m}(M) + \Psi_b^{-\infty,\gamma}(M).$$

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We can recursively improve the parametrix and get stronger results if we add one additional piece, $\Psi^{-\infty,\gamma}(M)$ (no b!), which consists of smooth kernels with expansions at lb and rb lifted from M^2. Then we get, eg:

Theorem

If P is an elliptic b-operator on M then every eigenfunction of P in any weighted Sobolev space has a complete expansion in powers and powers of logs of x at ∂M.

The End!

There are many more things that can be done with the b-calculus and its friends, and many problems left for you to solve!