Dispersive estimates for wave equations with time-periodic dissipation

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1 Introduction

If one studies the Cauchy problem for

$$u_{tt} - a^2(t)\Delta u + 2b(t)u_t + m^2(t)u = 0, \qquad u(0,\cdot) = u_1,\ D_t u(0,\cdot) = u_2, \quad (1.1)$$

and investigates the long-time behaviour of solutions oscillations in coefficients may have a devastating influence on results. One may think of the following two particular examples

$$u_{tt} - a^2(t)\Delta u = 0, \qquad u(0,\cdot) = u_1,\ D_t u(0,\cdot) = u_2, \quad (1.2)$$

$$u_{tt} - \Delta u + m^2(t)u = 0, \qquad u(0,\cdot) = u_1,\ D_t u(0,\cdot) = u_2, \quad (1.3)$$

with non-constant periodic $a(t) \geq c > 0$ or $m(t)$. In both cases a parametric resonance phenomenon occurs, which gives rise to exponentially increasing solutions (Floquet effect).

Even if $a(t) = 2 + \sin(\log(e+t)^\alpha)$, $\alpha > 2$, the energy $\mathbb{E}(u;t) = \|\nabla u\|_2^2 + \|u_t\|_2^2$ satisfies

$$\sup_t \frac{\log \mathbb{E}(u;t)}{\log t} = \infty \tag{1.4}$$

for some data (Reissig-Smith, modified Floquet).

Much attention has been paid to sharp conditions on coefficients to avoid

(a) this exponential behaviour of the energy or much stronger

(b) to retain decay estimates in a uniform sense.

One typical result is the following is as follows:

Theorem 1.1. *(Reissig-Smith, 2005) Consider the Cauchy problem for (1.2) and assume that $a(t)$ satisfies $0 < c_1 \le a(t) \le c_2$ and*

$$\left|\frac{\mathrm{d}^k}{\mathrm{d}t^k}a(t)\right| \le C_k(1+t)^{-k}\big(\log(e+t)\big)^{\gamma k} \tag{1.5}$$

for sufficiently many k. Then

$$\|u_t(t,\cdot), \nabla u(t,\cdot)\|_{L^q} \le C_{p,q,s}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+s}\|\langle \mathrm{D}\rangle u_1, u_2\|_{H^{p,r_p}} \tag{1.6}$$

holds true for any $p \in (1,2]$, $pq = p+q$ and $r_p = n(1/p - 1/q)$ with

- $s = 0$ *if* $\gamma = 0$,
- *any* $s > 0$ *if* $\gamma \in (0,1)$ *and*
- *any* $s > s_0$ *if* $\gamma = 1$, *where* s_0 *depends on* $a(t)$.

Furthermore, for any $\gamma > 1$ there exists a function $a(t)$ subject to (1.5) such that (1.4) $\sup_t \log \mathbb{E}(u;t)/\log t = \infty$ is valid.

The requirement (1.5) is rather strong, but sharp as the theorem states. If we just look for the energy, the situation becomes slightly better.

Theorem 1.2. *(Hirosawa, 2007) Consider the Cauchy problem for* (1.2) *and assume that* $a(t)$ *satisfies* $0 < c_1 \le a(t) \le c_2$ *and*

$$\left|\frac{\mathrm{d}^k}{\mathrm{d}t^k}a(t)\right| \le C_k(1+t)^{-qk}, \quad \textit{and} \quad \int_0^t |a(s)-a_\infty|\mathrm{d}s \le C(1+t)^p \qquad (1.7)$$

with $p, q \in [0,1)$ *and all* $k \le m$. *If additionally* $q \ge p + (1-p)/m$

$$c^{-1}\mathbb{E}(u;0) \le \mathbb{E}(u;t) \le c\mathbb{E}(u;0) \qquad (1.8)$$

holds true. If $q < p + (1-p)/m$, *then there exists a coefficient* $a(t)$ *subject to* (1.7) *and a sequence* u_j *of solutions with* $E(u_j;0) = 1$ *such that*

$$\sup_{t,j} \frac{\log \mathbb{E}(u_j;t)}{\log t} = \infty. \qquad (1.9)$$

A similar statement for a mass term $m^2(t)$ holds in both situations.

We want to discuss the problem

$$u_{tt} - \Delta u + 2b(t)u_t = 0, \qquad u(0,\cdot) = u_1,\ \mathrm{D}_t u(0,\cdot) = u_2, \tag{1.10}$$

for a time-dependent dissipation term and show that the situation is totally different. Now we are allowed to take periodic coefficients and still get good estimates:

Theorem 1.3. *(Wirth, 2007) Consider the Cauchy problem* (1.10) *with a.e. positive and periodic* $b(t) \in BV_{loc}$ *. Then the a-priori estimates*

$$\|u(t,\cdot)\|_{L^q} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|\langle \mathrm{D}\rangle u_1, u_2\|_{H^{p,r_p-1}}, \tag{1.11}$$

$$\|\nabla u(t,\cdot)\|_{L^q} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|\langle \mathrm{D}\rangle u_1, u_2\|_{H^{p,r_p}}, \tag{1.12}$$

$$\|u_t(t,\cdot)\|_{L^q} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-1} \|\langle \mathrm{D}\rangle u_1, u_2\|_{H^{p,r_p}}, \tag{1.13}$$

hold true for all $1 \le p \le 2 \le q \le \infty$ *with* $r_p > n(1/p - 1/q)$*.*

Note especially that the result follows for coefficients of very low regularity.

2 Treatment of the periodic problem

We study

$$u_{tt} - \Delta u + 2b(t)u_t = 0, \qquad u(0,\cdot) = u_1, \quad \mathrm{D}_t u(0,\cdot) = u_2, \tag{2.1}$$

$u_i \in \mathcal{S}'(\mathbb{R}^n)$, where $b(t)$ is continuous, of locally bounded variation and $b(t) > 0$ almost everywhere together with $b(t+T) = b(t)$ for some $T > 0$. After applying Fourier transform in x we obtain

$$\hat{u}_{tt} + |\xi|^2 \hat{u} + 2b(t)\hat{u}_t = 0, \tag{2.2}$$

which can be written as first order system

$$\mathrm{D}_t \begin{pmatrix} |\xi|\hat{u} \\ \mathrm{D}_t \hat{u} \end{pmatrix} = \begin{pmatrix} & |\xi| \\ |\xi| & 2\mathrm{i}b(t) \end{pmatrix} \begin{pmatrix} |\xi|\hat{u} \\ \mathrm{D}_t \hat{u} \end{pmatrix}. \tag{2.3}$$

2.1 The monodromy operator

We denote the coefficient matrix as $A(t,\xi)$ and solve the matrix-valued problem

$$\mathrm{D}_t \mathcal{E}(t,s,\xi) = A(t,\xi)\mathcal{E}(t,s,\xi), \qquad \mathcal{E}(s,s,\xi) = \mathrm{I} \in \mathbb{C}^{2\times 2}. \tag{2.4}$$

Then $(|\xi|\hat{u}, \mathrm{D}_t\hat{u})^T = \mathcal{E}(t,0,\xi)(|\xi|\hat{u}_1, \hat{u}_2)^T$. Since $b(t) \geq 0$ we immediately get $\|\mathcal{E}(t,s,\xi)\| \leq 1$, $t \geq s$. Furthermore, $\mathcal{E}(t+T, s+T, \xi) = \mathcal{E}(t,s,\xi)$ implies

$$\begin{aligned} \mathcal{E}(t+kT,s,\xi) &= \mathcal{E}(t+kT, t+(k-1)T,\xi)\cdots\mathcal{E}(t+T,t,\xi)\mathcal{E}(t,s,\xi) \\ &= \mathfrak{M}^k(t,\xi)\mathcal{E}(t,s,\xi), \end{aligned} \tag{2.5}$$

where

$$\mathfrak{M}(t,\xi) = \mathcal{E}(T+t,t,\xi), \qquad t \in \mathbb{R}, \tag{2.6}$$

denotes the family of monodromy operators associated to our problem. Results depend on spectral properties of these operators.

Proposition 2.1. *(i)* $\operatorname{spec}\mathfrak{M}(t,\xi)$ *is independent of* t.

(ii) The spectral radius satisfies $\rho(\mathfrak{M}(t,\xi)) < 1$ *for all* $\xi \neq 0$.

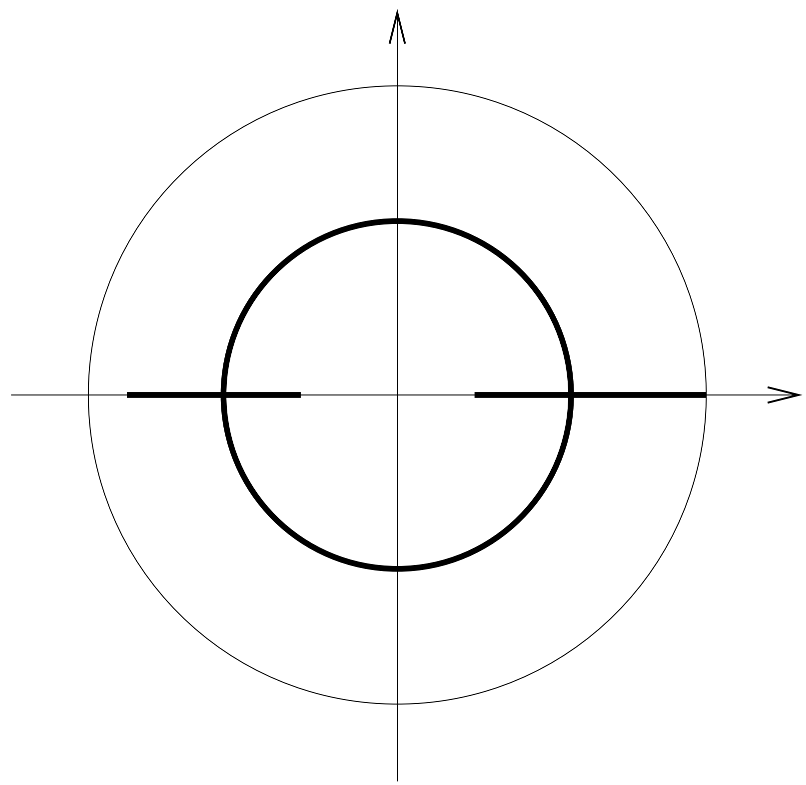
(iii) $\sup_{|\xi|\geq N} \|\mathfrak{M}(t,\xi)\| < 1$ *for* N *large enough.*

(iv) $1 \in \operatorname{spec}\mathfrak{M}(t,0)$.

Ideas of proof:

- eigenvalues of $\mathfrak{M}(t,\xi)$ are either real or complex-conjugate;
- $\det\mathfrak{M}(t,\xi) = \exp(-2\beta T)$, where $\beta = \frac{1}{T}\int_0^T b(s)\mathrm{d}s$;
- for fixed $\xi \neq 0$ we can exclude eigenvalues ± 1 (which would give $2T$-periodic solutions in contradiction to $b(t) > 0$ a.e.);
- as $|\xi| \to \infty$ asymptotic analysis provides estimates for $\|\mathfrak{M}(t,\xi)\|$.

Consequence: The spectrum of $\mathfrak{M}(t, \mathrm{D})$ looks like



and (for) us the only point of interest is $\zeta = 1$, which corresponds to $\xi = 0$. For $|\xi| \geq \epsilon$ the matrices $\mathfrak{M}(t, \xi)$ have (uniformly) contractive powers and we obtain *exponentially* decaying components of the solution.

2.2 The neighbourhood of $\xi = 0$

We have to be a bit more careful here. The point $|\xi|^2 = 0$ is interior point of the first instability interval of the corresponding *Hill's equation*

$$v_{tt} + \big((|\xi|^2 - b^2(t) - b'(t)\big)v = 0, \tag{2.7}$$

where

$$v(t) = \lambda(t)\hat{u}, \qquad \text{and} \qquad \lambda(t) = \exp\int_0^t b(s)\mathrm{d}s. \tag{2.8}$$

Therefore, in a neighbourhood of $|\xi|^2 = 0$ we can represent a fundamental system of solutions to (2.2) by Floquet's theorem.

Proposition 2.2. *(Floquet) A fundamental system of solutions to* (2.2) *is given by*

$$\mathrm{e}^{-\nu_\pm(\xi)t} f_\pm(t,\xi), \tag{2.9}$$

where $\exp(-\nu_\pm(\xi)T) \in \operatorname{spec} \mathfrak{M}(t,\xi)$, *denoted such that* $\nu_+(0) = 0$, *and*

- $\nu_\pm(\xi)$ *is real-valued and analytic in* $|\xi|^2$ *near* $|\xi|^2 = 0$;
- $f_\pm(t,\xi)$ *is* T*-periodic and differentiable;*
- $f_\pm(t,\xi)$ *is analytic in* $|\xi|^2$ *near* $|\xi|^2 = 0$.

Furthermore, $f_+(t,0) = 1$ *and* $f_\pm(t,\xi) \neq 0$ *for all* t *and small* $|\xi|$.

Therefore, near $\xi = 0$ we have a representation of the form

$$\hat{u}(t,\xi) = \sum_{j=1,2} \sum_{\pm} C_{j,\pm}(\xi) \mathrm{e}^{-\nu_\pm(\xi)t} f_\pm(t,\xi) \hat{u}_j(\xi) \tag{2.10}$$

with coefficients $C_{j,\pm}(\xi)$ analytic in $|\xi|^2$. All $-$ terms decay exponentially (because $\nu_-(0) = -2\beta T < 0$). For the $+$ terms we know $C_{+,j}(0) \neq 0$ together with

$$\nu_+(\xi) = -\sum_{k=1}^{\infty} \alpha_k |\xi|^{2k}, \qquad \alpha_1 > 0 \tag{2.11}$$

and

$$f_+(t,\xi) = 1 + \sum_{k=1}^{\infty} \gamma_k(t) |\xi|^{2k}, \qquad \gamma_k(t) \in C_T^1(\mathbb{R}). \tag{2.12}$$

Both series are uniformly convergent.

Consequence: Solutions look (modulo terms decaying faster) like

$$\hat{u}(t,\xi) \approx \sum_{j=1,2} C_{j,+}(0)\mathrm{e}^{-\alpha_1|\xi|^2 t}\hat{u}_j(\xi), \tag{2.13}$$

$$\partial_t\hat{u}(t,\xi) \approx \sum_{j=1,2} C_{j,+}(0)\mathrm{e}^{-\alpha_1|\xi|^2 t}\big(\gamma_1(t)-\alpha_1\big)|\xi|^2\hat{u}_j(\xi), \tag{2.14}$$

(provided that $\gamma_1(t) \neq \alpha_1$ or $\sum_j C_{j,+}(0)\hat{u}_j(0) \neq 0$, which would eliminate these main terms and therefore give a faster decay).

Now the estimates of Theorem 1.3 follow from mapping properties of the Fourier transform $\mathcal{F} : L^p \to L^{p'}$, $pp' = p + p'$, and Hölder inequality.

2.3 Diffusion phenomenon

The representation of solutions looks parabolic near $\xi = 0$. Indeed, if we compare with solutions to

$$w_t = \alpha_1 \Delta w, \qquad w(0,x) = w_0, \tag{2.15}$$

we get

$$\hat{w}(t,\xi) = \mathrm{e}^{-\alpha_1 |\xi|^2 t} \hat{w}_0(\xi). \tag{2.16}$$

Therefore, the following statement holds true:

Theorem 2.3. *The solutions $u(t,x)$ of (2.1) and $w(t,x)$ of (2.15) with Cauchy data related via*

$$w_0 = C_{1,+}(0) u_1 + C_{2,+}(0) u_2 \tag{2.17}$$

satisfy the asymptotic relation

$$\|u(t,\cdot) - w(t,\cdot)\|_{\dot{H}^s} \lesssim (1+t)^{-1-\frac{s}{2}} \big(\|u_1\|_{H^{s+1}} + \|u_2\|_{H^s} \big). \tag{2.18}$$

Remarks:

1. It is essential to use the homogeneous norm on the left hand side, otherwise we would get no improved orders.

2. α_1 is *not* the mean value of $1/2b(t)$ as one might expect. However, the estimate $\alpha_1 \le (2\beta)^{-1}$ holds true (which in turn implies by the harmonic-arithmetic mean inequality that α_1 is the mean value if and only if $b(t)$ is constant).

3. If $\hat{w}_0(\xi)$ vanishes in the following weak sense in $\xi = 0$

$$|\xi|^{-\sigma}\hat{w}_0(\xi) \in L^2_{loc}(\mathbb{R}^n), \tag{2.19}$$

then the rates improve by the exponent $-\frac{\sigma}{2}$. Similar for the dispersive estimates (replacing $L^2_{loc}(\mathbb{R}^n)$ by $L^p_{loc}(\mathbb{R}^n)$).

3 Further remarks and open problems

1. If we look more generally to wave equations with time-dependent dissipation term

$$u_{tt} - \Delta u + 2b(t)u_t = 0 \tag{3.1}$$

then different cases occur. If $tb(t)$ is bounded, the dissipation is *non-effective* and regularity of the coefficient together with symbol like estimates is required to obtain dispersive estimates.
[J. Wirth, J. Differential Equations, 2006]

2. Energy estimates alone can be derived under weaker assumptions on derivatives in combination with a stabilisation condition.
[F. Hirosawa, J. Wirth, J. Math. Anal. Appl., to appear]

3. If $tb(t) \to \infty$ the dissipation is *effective*. In this case I think that one could prove estimates without assumptions on derivatives (except local BV).
   This is still an open problem.

4. If we additionally assume estimates for derivatives (with improvement t^{-1}) we get dispersive estimates and a diffusion phenomenon relating solutions between

$$u_{tt} - \Delta u + 2b(t)u_t = 0, \qquad \text{and} \qquad w_t = \frac{1}{2b(t)}\Delta w. \tag{3.2}$$

   [J. Wirth, J. Differential Equations, 2007]

   This parabolic reference equation is in general wrong, as our considerations for periodic coefficients show. It is still open to understand the reasons for that.

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