Bellman function method in Harmonic Analysis was introduced by Donald Burkholder for finding the norm in \( L^p \) of martingale transform. Later it became clear that scope of the method is relatively wide, in particular, by this method a necessary and sufficient condition for the two weight martingale transform to be bounded was obtained.

So far the Bellman function technique can be credited for helping to solve several old Harmonic Analysis problems and by unifying approach to many others. In the first category one would name the sharp weighted estimates of such classical operators as the Hilbert transform (Petermichl) and the Ahlfors-Beurling transform (Petermichl-Volberg). In the second category one can name all kind of dimension free estimates of weighted and unweighted Riesz transforms. More precisely, Bellman function approach turned out to be useful in estimating classical singular integrals and moreover, estimations come out in a streamlined, standard way. The illustration of this is, for example, how one of the Bellman functions can be used to give a new bilinear Littlewood-Paley estimate of very general nature. It is valid for Kato semigroup with real coefficients, and actually even for more general Laplacians. In particular, this bilinear Littlewood-Paley estimate will be valid for euclidean, Ornstein-Uhlenbeck, Hermite, and Laguerre semigroups. It is dimension free also. This gives a completely unified approach to dimension free \( L^p \) estimates of Riesz transforms for those semigroups. Previously most of these estimates were obtained by the array of different methods by many people: E.Stein, G.Pisier, P.A. Meyer, F. Lust-Piquard, J.-L. Torrea, A. Nowak, K. Stempak, C. Guitiérrez to name just a few.

Roughly, Bellman function method makes apparent the hidden convexity and hidden scaling properties of a given Harmonic Analysis problem. Conversely, given a Harmonic Analysis problem with certain scaling properties one can (formally) associate with it a non-linear PDE, the so-called stochastic Bellman equation of the problem.

Bellman function technique is amazingly winning material for graduate students. The distance between the actual Harmonic Analysis problem and sitting and computing concrete formulas and getting (may be partial) but concrete results is almost zero. The student can plunge into the real calculations (always non-trivial ones) almost immediately. However, up till recently it was a matter of luck and tenacity of the individuum to find a supersolution of this abovementioned Bellman PDE (let us call it temporarily a Bellman function) or better the solution of this Bellman PDE (let us call it temporarily the Bellman function). However, recently the Bellman function technique got features of a theory which can be learned and relatively widely applied.

Below is the plan of 4 lectures that introduce this theory.
1. Lectures 1 and 2

The first lecture is introductory, we described in it several problems in which Bellman function technique historically appeared. Namely, the Burkholder $L^p$ estimate of the martingale transform, two weight estimate of the martingale transform (the most baby version of it) and a couple of so-called Buckley’s inequalities (used by Fefferman, Kenig and Pipher in their study of elliptic measures).

In the second lecture we jump to stochastic point of view, formulate Bellman’s principle, use Ito’s formula, and derive stochastic Bellman PDE for a general problem of Stochastic Optimal Control.

2. Lectures 3 and 4

In the third lecture we combine what was done in Lectures 1 and 2. Namely, for examples used in Lecture 1 we write down (formally) the corresponding problem of Stochastic Optimal Control and its Bellman PDE. Then we find the supersolutions for these Bellman PDE for Buckley’s inequalities and for a simplest version of two weight martingale transform.

Fourth lecture is devoted to the explanation of Burkholder’s solution for $L^p$ estimates of martingale transform. We explain that the beautiful and comically looking Burkholder function can be obtained by the same receipt: set up a Stochastic Optimal Control for Burkholder’s problem, write (formally) its Bellman PDE, and solve it using intrinsic scaling of the problems. As a result, the Burkholder function surfaces on the blackboard at the end of the lecture. This method of writing down Burkholder’s function is different from the original method, and seems to be considerably more transparent.

If time permits, we plan to explain how one can approach the problem of the sharp estimate of martingales of Burkholder type, but with extra orthogonality property (an analog of Cauchy-Riemann equation, but for martingales). Quite unexpectedly, the Bellman function for that problem turned out to be related to zeros of classical orthogonal polynomials. In its turn orthogonal martingales estimate leads to improved estimates of Ahlfors–Beurling transform, whose importance was explained in the previous lectures.

The lectures do not require the knowledge of martingale theory. Only the most basic concepts of probability will be used. On the analysis side, the lectures do not require anything more than Hahn–Banach theorem. No special knowledge of singular integrals theory is assumed.