1 A general overview over the content of the School

Probably this text will be updated several times, suggestions are welcome

1.1 The locally symmetric spaces, the (orbi-)local systems and the cohomology groups

Arithmetic groups are groups of the form $\Gamma = \mathrm{Sl}_n(\mathbb{Z}), \mathrm{Sp}_g(\mathbb{Z})$... or more generally subgroups of finite index of those. They are per definitionem discrete subgroups of real Lie groups $G(\mathbb{R})$, for instance $\mathrm{Sl}_n(\mathbb{Z}) \subset \mathrm{Sl}_n(\mathbb{R})$. They act on the symmetric space $X = G(\mathbb{R})/K_\infty$, here K_∞ is a maximal compact subgroup, for example $K_\infty = \mathrm{SO}(n) \subset \mathrm{Sl}_n(\mathbb{R})$. The quotient spaces $\Gamma\backslash X$ are very interesting Riemannian manifolds (with possibly some "singularities").

It is a fundamental fact that these quotients are not compact in general, but there are several ways to compactify $\Gamma\backslash X \subset \Gamma\backslash\bar{X}$, the Borel-Serre compactification and its consequences will be discussed. We will discuss the "tubular" neighborhood $\overset{\bullet}{\mathcal{N}}\ \partial(\Gamma\backslash\bar{X})$.

We introduce sheaves $\tilde{\mathcal{M}}$ with values in finitely generated abelian groups, which are obtained from finitely generated Γ-modules $\mathcal{M}$. We are interested in the cohomology groups $H^q(\Gamma\backslash X, \tilde{\mathcal{M}})$. They can be computed as the cohomology groups of a suitable Čzech complex. (See chap.2, 2.1.7) (Some basic knowledge of sheaf cohomology is assumed at this point).

The case that

$$\Gamma = \mathrm{Sl}_2(\mathbb{Z}), \mathbb{H} = \mathrm{Sl}_2(\mathbb{R})/\mathrm{SO}(2) = \text{ the upper half plane,}$$

and then the Γ module

$$\mathcal{M}_n = \{\sum_{\nu=0}^{\nu=n} a_\nu X^\nu Y^{n-\nu}, a_\nu \in \mathbb{Z}\}$$

is an avatar of such an object. In this case we can give an explicit expression of $H^1(\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}, \tilde{\mathcal{M}}_n)$.

By a general theorem of Raghunathan these cohomology groups are finitely generated abelian groups.

1.2 The finer structure of the cohomology groups

a) The cohomology groups $H^q(\Gamma\backslash X, \tilde{\mathcal{M}})$ have a filtration which is induced by the non-compactness of $\Gamma\backslash X$. We have the fundamental exact sequence

$$H^{i-1}(\partial(\Gamma\backslash\bar{X}), \tilde{\mathcal{M}}_{\mathbb{Z}}) \;\to\; H^i_c(\ \partial(\Gamma\backslash\bar{X}), \tilde{\mathcal{M}}_{\mathbb{Z}}) \;\to\; H^i(\partial(\Gamma\backslash\bar{X}), \tilde{\mathcal{M}}_{\mathbb{Z}}) \;\xrightarrow{res}\; H^i(\partial(\Gamma\backslash\bar{X}), \tilde{\mathcal{M}}_{\mathbb{Z}}) \;\to$$

$$p_c \searrow \qquad H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}}) \qquad \nearrow q_!$$

b) These cohomology groups are modules for the so called Hecke algebra $\mathcal{H}$. (See below) This Hecke algebra contains a central subalgebra (the unramified Hecke algebra $\mathcal{H}_{\text{un}}$) which is generated by Hecke operators $T_p^{(\chi)}$. (Here p runs over all "unramified" primes (depending on the choice of Γ there is a finite set of "ramified" primes) and χ runs over a finite set of cocharacters.)

c) For any commutative ring R with identity we can consider the sheaf $\tilde{\mathcal{M}}_R = \tilde{\mathcal{M}} \otimes R$ and we can define the cohomology groups

$$H^q(\Gamma\backslash X, \tilde{\mathcal{M}}_R).$$

They still have the above filtration and are modules for the Hecke algebra.

The fundamental exact sequence is an exact sequence of modules for the Hecke algebra.

We will give an explicit description of the fundamental exact sequence in the case

$$H^1_c(\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}, \tilde{\mathcal{M}}_n) \to H^1(\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}, \tilde{\mathcal{M}}_n) \to H^1(\partial(\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}), \tilde{\mathcal{M}}_n)$$

1.3 The Hecke algebra, the Satake isomorphism

The Hecke algebra is a restricted tensor product $\mathcal{H} = \bigotimes' \mathcal{H}_p$. At an unramified prime p the local Hecke algebra is described by the Satake isomorphism. Satake's theorem asserts says that the homomorphisms to a field F, i.e. $\pi_p : \mathcal{H}_{\text{un}} \to F$ are in one to one correspondence to conjugacy classes $\omega(\pi_p) \in G^\vee(F)$ where $G^\vee$ is the Langlands dual group. In the case that the underlying group is GL_n the "Satake parameter" attached to a $\pi_p : \mathcal{H}_{\text{un}} \to F$ is a diagonal element

$$\omega(\pi_p) = \begin{pmatrix} \omega_{1,p} & 0 & 0\dots & 0 & \\ 0 & \omega_{2,p} & 0 & \dots & \\ & & \ddots & & \\ 0 & \dots & & \dots & 0 \\ 0 & \dots & 0 & 0 & \omega_{n,p} \end{pmatrix} \in \mathrm{GL}_n(\bar{F})$$

whose conjugacy class is invariant under the Galois group $\mathrm{Gal}(\bar{F}/F)$.

This allows us to consider "eigenspaces"

$$H^q(\Gamma\backslash X, \tilde{\mathcal{M}}_F)(\pi_f) = \{x \in H^q(\Gamma\backslash X, \tilde{\mathcal{M}}_F) | T_p^{(\chi)}(x) = \pi_f(T_p^{(\chi)})x\}$$

where $\pi_f : \mathcal{H}_{\text{un}} \to \mathcal{O}_F$ is a homomorphism and $\mathcal{O}_F$ is the ring of algebraic integers of a number field F. Then $\pi_f = \otimes\pi_p$.

To these eigenspaces we can attach the so called cohomological L-functions

$$L^{\text{coh}}(\pi_f, r, s) = \prod_{p:prime} L^{\text{coh}}(\pi_p, r, s).$$

Up to here the discussion is on a purely combinatorial level, the cohomology groups and the action of the Hecke operators can be computed from the Czech complex of a suitable finite acyclic covering of $\Gamma\backslash X$ by open sets.

As an example we discuss the again the case $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$. In this simple case we can formulate the general problem of computing the *denominator of the Eisenstein class* .

The determination of this denominator (it is the numerator of $\zeta(-1-n)$*) is the first main result of these lectures.*

(First section of lectures in the School. About 4-5 lectures (Harder, Kaiser)

For a deeper understanding of these L-functions we need tools from analysis (the theory of automorphic forms, representation theory and other things).

Under certain conditions these tools allow us to prove that these functions $L^{\mathrm{coh}}(\pi_f, r, s)$ are meromorphic (or even holomorphic) functions in the variable s and they satisfy a functional equation. (Whittaker models and L-functions). The Riemann ζ-function shows up in this context.

We will discuss the influence of the analytic properties of these L- functions on the structure of the cohomology groups (Eisenstein cohomology). The cohomological interpretation of these L-functions provides some rationality results for these L-functions at special arguments. Then we will investigate the influence of these values at special arguments on the structure of the cohomology groups $H^q(\Gamma\backslash X, \tilde{\mathcal{M}})$. (Second and third section of lectures in the school)

Again under some assumptions we can attach representations of the Galois group $\rho(\pi_f) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(M(\pi_f))$ to such eigenspaces. Eventually we will see that values of $L(\pi_f, r, s_0)$ at certain specific arguments have influence on the structure of the Galois group. (Last section of lectures)

2 Some more detailed explanations

2.1 The L-functions

After tensoring by a suitable algebraic number field $F \subset \mathbb{C}$ we may write a filtration

$$H^q(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes F) \supset \mathcal{F}^1 H^q(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes F) \supset \dots \mathcal{F}^\nu H^q(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes F) \dots \supset \{0\}$$

by $\mathcal{H}$ invariant subspaces such that the successive quotients

$$F^\nu H^q(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes F)/F^{\nu+1} H^q(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes F)$$

are absolutely irreducible $\mathcal{H}$ modules $H^q(\pi_{\nu,f})$. These subquotients are restricted tensor products

$$H^q(\pi_f) = \bigotimes_{p:prime}{}' H(\pi_p)$$

where $H(\pi_p)$ are absolutely irreducible $\mathcal{H}_p$ modules. At the unramified places such absolutely irreducible $\mathcal{H}_p$ modules are one dimensional and π_p is simply a homomorphism $\pi_p : \mathcal{H}_p \to \mathcal{O}_F$, hence they are determined by their values $\pi_f(T_p^{(i)})$.

To such an irreducible $\mathcal{H}$ module $H(\pi)$ and a second parameter r (an irreducible representation of the Langlands dual group) we can attach a (cohomological) L-function

$$L^{\mathrm{coh}}(\pi_f, r, s) = \prod_{p:prime} L^{\mathrm{coh}}(\pi_p, r, s)$$

where the local factors are of the form

$$L^{\mathrm{coh}}(\pi_p, r, s) = (1 - A_1(\pi_p, r)p^{-s} + \ldots A_d(\pi_p, r)p^{-ds})^{-1}$$

and where the $A_i(\pi_p, r) \in \mathcal{O}_F$, for unramified $\mathcal{H}_p$ the $A_i(\pi_p, r)$ are certain expressions in the $\pi(T_p^{(i)})$.

The products are convergent for $\Re(s) >> 0$. The Riemann $\zeta-$function occurs in this family of L-functions.

2.2 Relation to automorphic forms, the computation by transcendental methods.

The cohomology groups $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes \mathbb{C})$(the $^\bullet$ means we look at all degrees simultaneously) are related - via the Eichler-Shimura isomorphism - to automorphic forms. In other words we can apply analytic tools to get insight into the structure of the cohomology. For instance in certain cases certain subspaces of $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes \mathbb{C})$ can be identified with spaces of holomorphic modular forms. The central tool is the identification

$$\Omega^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_{\mathbb{C}}) = \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{R})) \otimes \mathcal{M}_C) \qquad (1)$$

We can use the theory of representations of $G(\mathbb{R})$ and Hodge-theoretic type of argument to prove the semi-simplicity of the so called "inner cohomology" under the action of the Hecke algebra, in certain cases we get formulas for the multiplicities $m(\pi)$ (multiplicity one theorems).

Finally the analytic theory of automorphic forms provides instruments for an understanding of the analytic properties of the L-functions $L(\pi, r, s)$ as functions in the variable s. Under certain assumptions it is possible to show that $L(\pi, r, s)$ extends to a meromorphic (or even holomorphic) function in the entire s-plane and satisfies a functional equation. (Second series of lectures: 2-3 lectures Schwermer, Harder, Kaiser)

2.3 Eisenstein cohomology and special values

The spaces $\Gamma\backslash X$ are not compact in general, reduction theory tells us that we can describe some neighborhood $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)$ of infinity such that its complement in $\Gamma\backslash X$ is compact. This neighborhood of infinity is a union of open subsets

$$\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X) = \bigcup_P \overset{\bullet}{\mathcal{N}}_P(\Gamma\backslash X)$$

where P runs over the Γ- conjugacy classes of proper parabolic subgroups. We describe these pieces $\overset{\bullet}{\mathcal{N}}_P(\Gamma\backslash X)$ and their intersections in terms of fiber bundles over locally symmetric domains $\Gamma_H\backslash X^H$ attached to smaller reductive groups. This allows to compute the cohomology $H^\bullet(\overset{\bullet}{\mathcal{N}}_P\,(\Gamma\backslash X))$ in terms of the cohomology groups

$$H^\bullet(\Gamma_H\backslash X^H, \tilde{\mathcal{M}}(w))$$

where $\mathcal{M}(w)$ is a collection of Γ_H modules labeled by certain elements w in the Weyl group.

The goal of Eisenstein cohomology is to understand the restriction map

$$\ker(res) = H_!^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) \hookrightarrow H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) \xrightarrow{res} H^\bullet(\overset{\bullet}{\mathcal{N}}\,(\Gamma\backslash X), \tilde{\mathcal{M}}). \qquad (2)$$

The right hand side contains "pieces" $H(\sigma) \subset H^\bullet(\Gamma_H\backslash X^H, \tilde{\mathcal{M}}(w))(\sigma) \subset H^\bullet(\overset{\bullet}{\mathcal{N}}\,(\Gamma\backslash X), \tilde{\mathcal{M}})$ where σ is now an absolutely irreducible module for the Hecke algebra $\mathcal{H}_H$.

We want to understand how the "piece" $H(\sigma)$ is related to the image of res, for instance is $H(\sigma) \subset \mathrm{Im}(res)$?

a) We will explain that in certain situation the answer depends on w and whether a certain monomial expression

$$\mathcal{L}(\sigma, s) = \prod_r \frac{L(\sigma, r, m_r s)}{L(\sigma, r, m_r s + 1)}$$

is holomorphic at $s = 0$ or it has a pole.

b) We encounter situations where $\mathcal{L}(\sigma, s)$ is holomorphic and the cohomological interpretation yields a rationality result for special values of L-functions

$$\frac{1}{\Omega(\sigma)}\mathcal{L}(\sigma, 0) \in F^\times. \qquad (3)$$

Here $\Omega(\sigma) \in \mathbb{C}^\times$ is a "period" which is well defined up to a unit in $\mathcal{O}_F^\times$ and which is obtained from the comparison of two different descriptions of the cohomology.

c) Once we have such a rationality result we may ask: What do these numbers tell us? We formulate some conjectures which roughly say that these have "influence" on the structure of the restriction map (2) considered as map between $\mathcal{H}$-modules. A somewhat very optimistic statement would be: If some power $\mathfrak{p}^{n_\mathfrak{p}}$ occurs in the prime factorization of the denominator of $\frac{1}{\Omega(\sigma)}\mathcal{L}(\sigma, 0)$ then we should find elements of order $\mathfrak{p}^{n_\mathfrak{p}}$ in

$$\xi(\sigma) \in \mathrm{Ext}^1_{\mathcal{H}}(H^\bullet(\overset{\bullet}{\mathcal{N}}\,(\Gamma\backslash X), \tilde{\mathcal{M}})(\sigma) \otimes \mathcal{O}_F/\mathfrak{p}^{n_\mathfrak{p}}, H_!^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes \mathcal{O}_F/\mathfrak{p}^{n_\mathfrak{p}}) \qquad (4)$$

This assertion is stronger than the statement that the Eisenstein class $\mathrm{Eis}(\sigma)$ has denominator $\mathfrak{p}^{n_\mathfrak{p}}$. We will scetch the proof

Third series 2-3 lectures Harder-Schwermer -Kaiser.

2.4 Galois-modules

There is a general idea, which goes back to several people and summarized under the name "Langlands philosophy", that to such an isotypical piece $H(\pi_f)$ should correspond a collection of "motives" $M(\pi, r)$ such that we have an equality of L-functions

$$L^{\text{coh}}(\pi_f, r, s) = L(M(\pi, r), s) \tag{5}$$

This then implies that we should be able to attach to π (and some standard choice $r = r_0$ for the representation of the dual group) a compatible system of Galois representations

$$\{\rho_\ell : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_{d(\pi)}(\mathbb{Q}_\ell)\} \tag{6}$$

such that we get an equality of L-functions. The existence of such a compatible system of Galois-modules has been proved in many cases.

The following case will be discussed in some detail. We assume that the quotient $\Gamma\backslash X$ is a Shimura variety, we even assume that $\Gamma\backslash X$ is in a certain natural way the set of complex valued points of a quasi projective variety $S/\text{Spec}(\mathbb{Q})$. For any prime ℓ we can interpret $\mathcal{M} \otimes \mathbb{Z}_\ell$ as a sheaf for the etale topology on S and we can consider the etale cohomology groups

$$H^\bullet_{\text{ét}}(S \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_\ell) = \lim_{\leftarrow} H^\bullet_{et}(S \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell^m.\mathbb{Z}),$$

These cohomology groups are Galois-modules, and we have some control on ramification.

We may also read this as follows: We have the comparison isomorphism

$$H^\bullet_{et}(S \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell^m\mathbb{Z}) \xrightarrow{\sim} H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell^m\mathbb{Z})$$

and we use this to put the structure of a Galois module on $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell^m\mathbb{Z})$, the Galois module structure commutes with the action of Hecke operators.

We can compactify $S \to S^\vee$ and this gives us a Galois-module structure on $H^\bullet(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell^m\mathbb{Z})$ as well and finally we get a $\mathcal{H} \times \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ invariant homomorphism

$$H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell^m\mathbb{Z}) \xrightarrow{res} H^\bullet(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell^m\mathbb{Z}). \tag{7}$$

Hence we get from the extension classes in $\xi(\sigma)$ in (4) interesting extension of Galois-modules. (The prime ideal $\mathfrak{p}$ is now ℓ).

This will be the last series of 2-3 lectures : E. Hellmann- P. Scholze

Here we discuss a very old theme in number theory: How do special values of L functions have influence on the structure of the Galois group.