Variational Methods for the Analysis of Patterns

1 Background knowledge

1. **Sobolev spaces**

   a) Show that for any $u \in H^1(\mathbb{R})$,
   $$|u(x) - u(y)| \le \|u\|_{H^1(\mathbb{R})}|x-y|^{1/2}.$$

   b) Show that $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, and that there exists $C > 0$ such that
   $$\|u\|_{L^\infty(\mathbb{R})} \le C\|u\|_{H^1(\mathbb{R})}.$$

   c) Let $u \in H^1(\mathbb{R})$, and let $f \in C^1(\mathbb{R})$ with $f(0) = 0$. Show that $f \circ u \in H^1(\mathbb{R})$, and that
   $$(f \circ u)'(x) = f'(u(x))\, u'(x)$$
   in the sense of weak derivatives.

   d) Let $u \in H^1(-1,0)$ and $v \in H^1(0,1)$, and define the composite function
   $$w(x) := \begin{cases} u(x) & \text{if } -1 < x < 0 \\ v(x) & \text{if } 0 < x < 1 \end{cases}$$
   Show that w is an element of $H^1(-1,1)$ if and only if $u(0) = v(0)$.

   e) Deduce the corresponding characterization for 'gluing' elements of $H^2(-1,0)$ and $H^2(0,1)$ to an element of $H^2(-1,1)$.

   f) Let $\Omega \subset \mathbb{R}$. Show that the norms
   $$\|u\|_a^2 := \int_\Omega \Big[u''^2 + u'^2 + u^2\Big] \qquad \text{and} \qquad \|u\|_b^2 := \int_\Omega \Big[u''^2 + u^2\Big]$$
   are equivalent.

2. **Weak convergence**

   a) Let H be a Hilbert space. Show that if $u_n \to u$ in H and $v_n \rightharpoonup v$ in H, then
   $$(u_n, v_n) \to (u, v) \qquad \text{as } n \to \infty.$$

   b) Give an example of a sequence of functions $u_n : (0,1) \to \mathbb{R}$ such that
   - $u_n \rightharpoonup u$ in $L^2(0,1)$
   - $u_n^2 \rightharpoonup v$ in $L^1(0,1)$

   and $v \ne u^2$.

   c) Let H be a real Hilbert space. Show that if $u_n \rightharpoonup u$ in H, and $\|u_n\| \to \|u\|$, then in fact $u_n \to u$ in H.

3. **Weak solutions of elliptic PDEs**

   a) Let Ω be a bounded open set in $\mathbb{R}^d$ with smooth boundary, and let $f \in L^2(\mathbb{R}^d)$. Write a weak formulation for the elliptic problem
   $$\begin{aligned} -\Delta u &= f \qquad && \Omega, \\ u &= 0 && \partial\Omega, \end{aligned}$$
   and show that there exists a solution in $H^1(\Omega)$. How is the boundary condition encoded? What is the corresponding minimization problem?

b) Address the same questions for the problems

$$
\begin{aligned}
\Delta^2 u &= f && \Omega,\\
u &= 0 && \partial\Omega,\\
\frac{\partial u}{\partial n} &= 0 && \partial\Omega,
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta^2 u &= f && \Omega,\\
u &= 0 && \partial\Omega,\\
\Delta u &= 0 && \partial\Omega.
\end{aligned}
$$

c) Again the same questions for the problems

$$
\begin{aligned}
\Delta^2 u &= f && \Omega,\\
u &= g && \partial\Omega,\\
\frac{\partial u}{\partial n} &= h && \partial\Omega,
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta^2 u &= f && \Omega,\\
\Delta u &= g && \partial\Omega,\\
\nabla\Delta u \cdot n &= h && \partial\Omega,
\end{aligned}
$$

where g and h are smooth functions defined on $\partial\Omega$, and n is the outward unit normal.

4. **Functionals and their derivatives**
Recall: if X is a normed space, then a functional $F : X \to \mathbb{R}$ is Fréchet differentiable at $x \in X$ if there exists a bounded linear functional $A \in X'$ such that

$$F(x+y) - F(x) - Ay = o(\|y\|_X) \qquad \text{as } y \to 0.$$

a) Let $\Omega \subset \mathbb{R}^d$ be bounded. For the following functionals, determine whether they are Fréchet differentiable; if so, prove this and calculate their derivative; if not, disprove the differentiability:

i. $F : H^1(\Omega) \to \mathbb{R}$, $F(u) = \|u\|^2_{H^1(\Omega)}$
ii. $F : H^1(\Omega) \to \mathbb{R}$, $F(u) = \|u\|^2_{L^2(\Omega)}$
iii. $F : L^2(\Omega) \to \mathbb{R}$, $F(u) = \|u\|^2_{L^2(\Omega)}$
iv. $F : L^2(\Omega) \to \mathbb{R}$, $F(u) = \|u\|^2_{H^1(\Omega)}$
v. $F : L^3(\Omega) \to \mathbb{R}$, $F(u) = \|u\|^3_{L^3(\Omega)}$
vi. $F : L^3(\Omega) \to \mathbb{R}$, $F(u) = \|u\|_{L^3(\Omega)}$
vii. Assume $v \in L^2(\Omega)$; $F : L^2(\Omega) \to \mathbb{R}$, $F(u) = (u, v)_{L^2(\Omega)}$
viii. $F : L^2(\Omega) \to \mathbb{R}$, $F(u) = \int_\omega u$ where $\omega \subset \Omega$ (the answer will depend on ω; can you give necessary and sufficient conditions?)

ix. $F : L^2(\Omega) \to \mathbb{R}$, $F(u) = \int_\Omega \sqrt{1+u^2}$

b) Fix $F \in C^2(\mathbb{R})$ with $F(0) = F'(0) = 0$ and define on $H^1(\mathbb{R})$ the functional

$$E(u) := \int_{\mathbb{R}} \Big[\frac{1}{2}u'^2 + F(u)\Big].$$

i. Show that $E(u) \in \mathbb{R}$ for all $u \in H^1(\mathbb{R})$.

ii. Determine the Fréchèt derivative $E'(u)$, and prove that for each u fixed it is a bounded linear form on $H^1(\mathbb{R})$.

iii. Prove that $E \in C^1(H^1(\mathbb{R}); (H^1(\mathbb{R}))')$, i.e. prove that the mapping $u \mapsto E'(u)$ is continuous from $H^1(\mathbb{R})$ to the dual of $H^1(\mathbb{R})$.

2 Fourth-order ODEs

1. Fix $p \in \mathbb{R}$ and $F \in C^1(\mathbb{R})$ such that $F(0) = F'(0) = 0$. Consider the energy functional $E : H^2(\mathbb{R}) \to \mathbb{R}$,

$$E(u) := \int_{\mathbb{R}} \Big[\frac{1}{2}u''^2 - \frac{p}{2}u'^2 + F(u)\Big].$$

a) Determine the Fréchèt derivative of E.

b) Let $u \in H^2(\mathbb{R})$ be a stationary point; determine the equation that u satisfies (the Euler-Lagrange equation).

c) Consider the evolution problem given by

$$u_t = -u_{xxxx} - pu_{xx} - F'(u), \tag{1}$$

writing subscripts for differentiation. Show that E decreases along solutions of (1).

2. (Rescaling) Define $E : H^2(\mathbb{R}) \to \mathbb{R}$ by

$$E(u) := \int_{\mathbb{R}} \Big[\frac{a}{2}u''^2 - \frac{b}{2}u'^2 + F(u)\Big],$$

where $a > 0$, $b \in \mathbb{R}$, $F(0) = F'(0) = 0$, and $F''(0) > 0$. Show that by scaling E, u, and x, this functional is 'equivalent' to a functional

$$\tilde{E}(u) := \int_{\mathbb{R}} \Big[\frac{1}{2}u''^2 - \frac{p}{2}u'^2 + \tilde{F}(u)\Big],$$

for some $p \in \mathbb{R}$, and such that $\tilde{F}(0) = \tilde{F}'(0) = 0$ and $\tilde{F}''(0) = 1$.

3. (Rescaling 2) Define $E : H^2(\mathbb{R}) \to \mathbb{R}$ by

$$E(u) := \int_{\mathbb{R}} \Big[\frac{a}{2}u''^2 - \frac{b}{2}u'^2 + \frac{c}{2}u^2 - \frac{d}{4}u^4 + \frac{e}{6}u^6\Big],$$

where $a, c, d, e > 0$ and $b \in \mathbb{R}$. Show that by scaling E, u, and x, this functional is equivalent to a functional

$$\tilde{E}(u) := \int_{\mathbb{R}} \Big[\frac{1}{2}u''^2 - \frac{p}{2}u'^2 + \frac{1}{2}u^2 - \frac{1}{4}u^4 + \frac{\alpha}{6}u^6\Big],$$

for some $p \in \mathbb{R}$ and some $\alpha > 0$.

4. (Convexity) Let $F(u) = \frac{1}{2}u^2 + G(u)$, with $G : \mathbb{R} \to \mathbb{R}$ convex, and $p \le 2$. Show that the functional

$$E(u) := \int_{\mathbb{R}} \Big[\frac{1}{2}u''^2 - \frac{p}{2}u'^2 + F(u)\Big]$$

is convex on $H^2(\mathbb{R})$. What is the global minimizer?

5. **(The linear problem on $\mathbb{R}$, energy approach)** Consider the quadratic energy

$$E(u) = \int_{\mathbb{R}} \Big[\frac{1}{2}u''^2 - \frac{p}{2}u'^2 + \frac{1}{2}u^2\Big],$$

with $p \geq 2$. Show that E has no minimizers when $p > 2$. What is the infimum when $p = 2$?

6. **(The linear problem on a bounded domain, energy approach)** Consider again the quadratic energy

$$E(u) = \int_0^L \Big[\frac{1}{2}u''^2 - \frac{p}{2}u'^2 + \frac{1}{2}u^2\Big],$$

with $p \in \mathbb{R}$, on the set $H^2_\#(0,L)$ of periodic H^2-functions on $(0,L)$. Show that there exists a critical $p_c \in \mathbb{R}$ such that

   a) If $p < p_c$, then $u \equiv 0$ is the unique global minimizer;

   b) If $p = p_c$, then $u \equiv 0$ is again globally minimizing, but there also exists at least one other global minimizer (bonus question: characterize completely the set of global minimizers);

   c) If $p > p_c$, then $\inf\{E(u); u \in H^2_\#(0,L)\} = -\infty$.

   Roughly sketch a graph of p_c as a function of the domain length L.

7. **(Nonlinearities that make E coercive)** In the course we showed that when $F(u) = u^2/2 + \alpha u^4/4$ with $\alpha > 0$, sublevel sets of E in $H^2_\#(0,L)$ are bounded for any $p \in \mathbb{R}$.

   a) Show that the same holds for any $F \in C(\mathbb{R})$ satisfying

$$\liminf_{|s|\to\infty} \frac{F(s)}{s^2} = \infty.$$

   b) Generlize the result to $H^2(0,L)$ (without periodicity constraint).

3 Mountain-pass structure

Let X be a Banach space.

Definition 1 *A function $E : X \to \mathbb{R}$ satiesfies the Palais-Smale condition (at level c) if each sequence $(u_n) \subset X$ with $E(u_n) \to c$ and $\|E'(u_n)\|_{X'} \to 0$ has a (strongly) convergent subsequence.*

1. We now will prove that the functional

$$E(u) := \int_{\mathbb{R}} \Big[\frac{1}{2}u''^2 - \frac{p}{2}u'^2 + \tilde{F}(u)\Big]$$

satisfies the Palais-Smale condition on $H^2_\#(0,L)$ for $p < 2$ and $F(u) = u^2/2 + \alpha u^4/4$, $\alpha \in \mathbb{R}$.

   a) Show that any Palais-Smale sequence has a weakly converging subsequence.

   b) Use the strong convergence of $E'(u_n)$ to show that the norm $\|u_n''\|_{L^2}$ converges.

   c) Conclude.

   d) Generalize: study the case $F(u) = u^2/2 + \alpha u^3/3$, $\alpha \in \mathbb{R}$.

   e) Generalize the result to the case of general F, satisfying (for instance)

$$\exists c, C > 0 : \qquad cs^4 \leq s^2 - sF'(s) \leq Cs^4$$

   (Hint: the inequalities above imply related inequalities for $F(s) - s^2/2$). What conditions on c and C do you need?

We will use the

Theorem 2 ***(Mountain-pass theorem)*** *Let $E \in C^1(X;\mathbb{R})$, $e \in X$, and set*

$$\Gamma := \{\gamma \in C^0([0,1];X) : \gamma(0) = 0, \gamma(1) = e\}.$$

Define

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} E(\gamma(t)).$$

If $c > \max\{E(0), E(e)\}$ then there exists a sequence $(u_n) \subset X$ such that $E(u_n) \to c$ and $\|E'(u_n)\|_{X'} \to 0$.

1. Apply the mountain-pass theorem to the functional

$$E(u) := \int_0^L \Big[\frac{1}{2}u''^2 - \frac{p}{2}u'^2 + \frac{1}{2}u^2 - \frac{1}{4}u^4\Big].$$

   Don't forget to prove that $E \in C^1$.

   Comment on the properties of the object that you find: its stability, its energy level; can you imagine an algorithm to find such a point? Can you imagine a minimization problem for which this point would be a local minimum?

2. **A counter-example to the Palais-Smale condition (Brezis-Nirenberg 1991)**
   Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = x^2 + (1 - x)^3 y^2$$

   Show that f has mountain-pass structure: a mountain range separating two valleys, with height at least 1. Show that f has only the critical value 0. Explain that therefore f does not satisfy the Palais-Smale condition, and that a sequence of approximately-stationary points generated by the Mountain-Pass theorem can not converge.

3. **A second-order problem**
   Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be bounded, and fix $1 < p < (d - 2)/(d + 2)$. Define the functional $E : H^1_0(\Omega) \to \mathbb{R}$,

$$E(u) := \int_\Omega \Big[\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\Big]$$

   Use the mountain-pass theorem to prove the existence of a non-zero stationary point u of E. Determine the equation that u satisfies.

4. **A nonlocal problem**
   Let $\mathbb{T}$ be the one-dimensional torus of unit length, i.e. $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let $\kappa \in L^\infty(\mathbb{T})$, with $\kappa \geq 0$ and $\int \kappa = 1$. Define the functional $E : L^1 \to \mathbb{R}$ by

$$E(u) := \int_\mathbb{T} \Big[f(u) - \alpha u\, \kappa * u\Big],$$

   where $f(s) = \sqrt{1 + s^2}$ and $\alpha < 1/2$. Show that there exists a non-zero stationary point of E.

4 Unbounded and free-length domains

1. In the course we proved that minimizers u of the problem

$$\inf\{E_L(u) : L > 0,\ u \in H^2_\#(0, L)\}$$

with

$$E_L(u) := \int_0^L \Big[\frac{1}{2}u''^2 - \frac{p}{2}u'^2 + F(u)\Big]$$

are identical between each consecutive pair of maxima (and the maxima are each at the same level).

a) Use the same idea to prove that u is even about every maximum or minimum.

b) Assume that F is even and prove that u is also odd about each zero.

2. Let u be a minimizer in the same problem as above. Use partial stretching — stretching the function u on part of the domain $(0, L)$ — to derive a differential equation satisfied by u. Compare this equation to the Euler-Lagrange equation: what is the relationship between the two?

3. (Research-level exercise!) How could you use properties of the function

$$L \mapsto \inf\{E_L(u) : u \in H^2_\#(0, L)\}$$

to prove the claim

> If $L > 0$ is sufficiently large, then the global minimizer u of E_L (for fixed L) is the same between each pair of consecutive maxima.

Which properties would be sufficient? What would you need to prove them?

4. (A priori L^∞ bounds) In this exercise we will prove that solutions of

$$u'''' + pu'' + u - u^3 + \alpha u^5 = 0, \qquad \alpha > 0, \tag{2}$$

can not be arbitrarily large in L^∞, even if the domain becomes large. One classical result we will use is the general principle that 'L^∞-bounds imply regularity' for uniformly elliptic equations, which in our case becomes

Lemma 1 *Let $L > 1$ and $p \le p_c(1) - \varepsilon$ for some $\varepsilon > 0$. There exists $C = C(\varepsilon) > 0$ (independent of L) such that any solution $u \in H^2(0, L)$ of*

$$u'''' + pu'' + u = f \in L^\infty$$

satisfies $\|u\|_{H^4(x,x+1)} \le C\|f\|_\infty$ for any $x \in (0, L-1)$.

In order to apply this we assume that $p < p_c(1)$.

a) Assume, to force a contradiction, that there exists a sequence of functions $u_n \in H^2(-n, n)$ with $\|u_n\|_\infty \to \infty$, each satisfying (2) on $(-n, n)$. Rescale u_n and x in such a way that the rescaled function v_n on the rescaled domain $(-\ell_n, \ell_n)$ satisfies $\|v_n\|_\infty = 1$ and

$$v_n'''' + \alpha v_n^5 = \text{the rest}.$$

b) Use the lemma above to deduce that v_n converges weakly in $H^4_{\mathrm{loc}}(\mathbb{R})$ to a limit v.

c) Conclude that the limit v satisfies $\|v\|_\infty = 1$ and

$$v'''' + \alpha v^5 = 0 \qquad \text{on } \mathbb{R}.$$

d) Show that such a v does not exist, by multiplying the equation with an appropriately truncated version of v and integrating (or, equivalently, by using a truncated version of v in the weak formulation).

5. (The linear problem, ODE approach) Consider the equation

$$u'''' + pu'' + u = 0 \qquad \text{on } \mathbb{R}.$$

Determine the general solution of this equation. Classify the behaviour at $\pm\infty$ in terms of p.

6. (The nearly-linear problem) Consider the equation

$$u'''' + pu'' + F'(u) = 0 \qquad \text{on } \mathbb{R},$$

where $F(u) = u^2/2 + O(u^4)$ (or $F'(u) = u = O(u^3)$). What does the previous result say about behaviour of solutions of this equation that converge to zero at $\pm\infty$?

7. (A constrained problem) Now consider the functionals $E, S : H^2(\mathbb{R}) \to \mathbb{R}$,

$$E(u) := \int_{\mathbb{R}} \Big[\frac{1}{2}u''^2 + F(u)\Big] \qquad \text{and} \qquad S(u) := \frac{1}{2}\int_{\mathbb{R}} u'^2.$$

Derive the Euler-Lagrange equation for the constrained problem

$$\inf \Big\{E(u) : u \in H^2(\mathbb{R}),\ S(u) = \text{prescribed}\Big\}.$$

5 *c*-optimal minimizers

For brevity we write $\ell(u)$ for the density of the energy, which we take to be

$$\ell(u) = \frac{1}{2}u''^2 - \frac{p}{2}u'^2 + F(u),$$

and we define for $u \in H^2_{\mathrm{loc}}(\mathbb{R})$

$$G(u) := \liminf_{L\to\infty} \frac{1}{L}\int_{-L/2}^{L/2} \ell(u).$$

1. Show that a c-optimal minimizer satisfies the usual Euler-Lagrange equation.
2. Show that any function $u \in H^2_{\mathrm{loc}}(\mathbb{R})$ that satisfies the second condition of c-optimality, i.e. such that for every $L > 0$ it achieves the minimum in

   $$\inf\left\{\frac{1}{L}\int_{-L/2}^{L/2} \ell(\tilde{u}) :\ (\tilde{u}, \tilde{u}') = (u, u') \text{ at } \pm L/2\right\},$$

   also satisfies the first condition:

   $$G(u) = \inf\Big\{G(\tilde{u}) : \tilde{u} \in H^2_{\mathrm{loc}}(\mathbb{R})\Big\}.$$

3. Define the two numbers

   $$\mu := \inf\Big\{G(\tilde{u}) : \tilde{u} \in H^2_{\mathrm{loc}}(\mathbb{R})\Big\}$$

   and

   $$\nu := \inf\{E_L(u) : L > 0,\ u \in H^2_{\#}(0, L)\}.$$

   Prove that $\mu = \nu$.

6 Modica-Mortola and the block copolymer energy

1. **Derivation of the Modica-Mortola functional from a convolution energy**
   In this exercise we study the connection between the Modica-Mortola functional

$$F_\varepsilon(x) = \int \Big[\varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\Big]$$

   for some double-well potential W and the functional

$$G_\varepsilon(x) = \int \Big[f(u) + \alpha(1-u)\,\kappa_\varepsilon{*}u\Big]$$

   where $\alpha > 0$, $f(u) = u\log u + (1-u)\log(1-u)$, and

$$\kappa_\varepsilon(x) = \varepsilon^{-d}\kappa(x\varepsilon^{-1}) \qquad \text{and} \qquad \kappa \geq 0,\ \kappa \text{ even, and } \int \kappa = 1.$$

   Here d is the space dimension.

   a) First take $\kappa = \delta$, the Dirac delta distribution; G_ε is now independent of ε. The behaviour of the functional G_ε depends on the value of α. Investigate the two cases, small and large α.

   b) Next take $\kappa \in L^1(\mathbb{R})$, $\kappa \geq 0$, even, and $\int \kappa = 1$. Derive for fixed smooth u an asymptotic development of $\kappa_\varepsilon{*}u$ up to order ε^2.

   c) Use this development in G_ε to derive a functional of the form F_ε. What form does W take?

2. **(Study of the H^{-1}-norm)**

   a) First consider the H^{-1}-norm on $\mathbb{R}^3$. Write for $u \in C_c(\mathbb{R}^3)$

$$\|u\|^2_{H^{-1}(\mathbb{R}^3)} := \frac{1}{4\pi}\iint \frac{u(x)u(y)}{|x-y|}\,dxdy$$

   Show that this number is finite for any $u \in C_c(\mathbb{R}^3)$.

   b) Compare the definition above with the alternative defintions

$$\|u\|^2_A := \int_{\mathbb{R}^3} |\nabla v|^2\,dy \qquad \text{where } -\Delta v = u, \quad \lim_{|x|\to\infty} v(x) = 0$$

$$\|u\|^2_B := \sup_{v\in C_c^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} [2uv - |\nabla v|^2]$$

   c) Now study the scaling: show that if we define for $u \in C_c(\mathbb{R}^3)$ the rescaled $u_\lambda(x) := u(x/\lambda)$, then

$$\|u_\lambda\|^2_{H^{-1}(\mathbb{R}^3)} = \lambda^5\|u\|^2_{H^{-1}(\mathbb{R}^3)}$$

   d) **(Tricky question)** How do the different characterizations above behave in one and two dimensions?

   e) Now switch to the three-dimensional torus $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$. How do the characterizations above change?

7 Gamma-convergence

1. **Equivalence of two definitions: for the fundamentalists**
Let X be a topological space. The lower and upper Gamma-limit of a sequence of functionals $F_n : X \to \mathbb{R}$ are defined as

$$(\Gamma - \liminf_{n\to\infty} F_n)(x) := \sup_{N\in\mathcal{N}(x)} \liminf_{n\to\infty} \inf_N F_n \quad \text{and} \quad (\Gamma - \limsup_{n\to\infty} F_n)(x) := \sup_{N\in\mathcal{N}(x)} \limsup_{n\to\infty} \inf_N F_n.$$

The Gamma-limit exists if the two coincide. The *sequential* Gamma-limit is defined as follows: F is the sequential Gamma-limit of F_n iff

- For each sequence $x_n \to x$ in X, $\liminf_{n\to\infty} F_n(x_n) \geq F(x)$;
- For each $x \in X$, there exists a sequence $x_n \to x$ such that $\limsup_{n\to\infty} F_n(x_n) \leq F(x)$.

Let the topology of X satisfy the first axiom of countability (i.e. each $x \in X$ has a countable local base of open sets). Show that the two definitions of Gamma-limit coincide: if one limit concept exists, then so does the other, and the two limits coincide. (Actually, it may be useful to know that the first countability condition is only necessary for one implication; which one?)

2. **Basic properties** (Assume the first axiom of countability if necessary or useful).
   - **a)** Show that uniform convergence implies Gamma-convergence.
   - **b)** Find an example of a sequence F_n such that $F_n \to F$ pointwise and $F_n \xrightarrow{\Gamma} G$, but $F \neq G$. Show that in this case always $G \leq F$.
   - **c)** Show that a subsequence of a Gamma-converging sequence converges to the same limit.
   - **d)** Let G be a fixed, continuous function; show that if $F_n \xrightarrow{\Gamma} F$, then $F_n + G \xrightarrow{\Gamma} F + G$. (Bonus question: can we generalize the fixed function G to a sequence G_n? What conditions on G_n are sufficient to again have $F_n + G_n \xrightarrow{\Gamma} F + G$? Can we assume $G_n \xrightarrow{\Gamma} G$?)
   - **e)** Let X be a topological vector space. Show that the Gamma-limit of convex functions is convex, the Gamma-limit of quadratic forms is quadratic, and for $\alpha > 0$ the Gamma-limit of α-homogenous forms is α-homogeneous.
   - **f)** Show that any Gamma-limit is lower semicontinuous.

3. **Concrete examples** These are taken from the book by Dal Maso.
Determine the pointwise and Gamma upper and lower limits (when they exist) of the following sequences of functions on $\mathbb{R}$.
   - **a)** $F_n(x) = nx \exp(-2n^2x^2)$
   - **b)**
   $$F_n(x) = \begin{cases} nxe^{-2n^2x^2} & \text{if } n \text{ is even} \\ 2nxe^{-2n^2x^2} & \text{if } n \text{ is odd} \end{cases}$$
   - **c)** $F_n(x) = nxe^{nx}$
   - **d)** $F_n(x) = \arctan nx$
   - **e)** $F_n(x) = \sin nx$
   - **f)** $F_n(x) = e^{-nx^2}$
   - **g)** $F_n(x) = -e^{-nx^2}$
   - **h)**
   $$F_n(x) = \begin{cases} 0 & \text{if } n(x - e^n) \text{ is integer} \\ 1 & \text{otherwise} \end{cases}$$