Dynamics of topological defects in nonlinear Hamiltonian PDEs.

Robert L. Jerrard

Department of Mathematics
University of Toronto

Hausdorff School on Nonlinear Evolutions
July 13-17, 2015
Hausdorff Center for Mathematics

topics for lectures

We will study limits $\varepsilon \to 0$ of equations

$$\left.\begin{matrix} - \\ u_t \\ u_{tt} \\ iu_t \end{matrix}\right\} - \Delta u + \frac{2}{\varepsilon^2} f(u) = 0.$$

- structural conditions on f allowing topological defects
- focus on hyperbolic, Schrödinger.
- elliptic and parabolic results are more classical.
- general question: **how do defects evolve?**.
- tools/estimates from calculus of variations (eg Γ-convergence).

(static) topological defects

Consider

$$E(u) := \int_{\mathbb{R}^k} \frac{1}{2}|\nabla u|^2 + F(u)\,dx, \qquad u \in H^1_{loc}(\mathbb{R}^k;\mathbb{R}^\ell)$$

for $F : \mathbb{R}^\ell \to \mathbb{R}$ such that

- $F \geq 0$, and $M := \{y \in \mathbb{R}^\ell : F(y) = 0\}$ is a submanifold of $\mathbb{R}^\ell$.
- $\pi_{k-1}(M)$ is nontrivial. For example:
  - $k = 1$ and M is disconnected, *e.g.* a finite set
  - $k = 2$ and M is not simply connected, *e.g.* $M \cong S^1$ or RP^2.
  - $k \geq 3$ and $M \cong S^{k-1}$.

a static defect: a (finite-energy) critical point u such that

$$u_\infty(x) := \lim_{r\to\infty} u(rx) \quad \text{exists for all } x \in S^{k-1}$$

and

$$u_\infty : S^{k-1} \to M \text{ is **not** homotopic to a constant}$$

static defects, continued

Example

$$E(u) := \int_{\mathbb{R}^k} \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(|u|^2 - 1)^2 \, dx, \qquad u \in H^1_{loc}(\mathbb{R}^k; \mathbb{R}^k)$$

for $k = 1$ or 2. Here

$$M = S^{k-1} = \{u \in \mathbb{R}^k : |u| = 1\}.$$

for $k = 1$, a defect solves

$$\boxed{-u'' + (u^2 - 1)u = 0, \qquad u(\pm\infty) = \pm 1}$$

This is a model *interface*

for $k = 2$, a defect solves

$$\boxed{-\Delta u + (|u|^2 - 1)u = 0, \qquad u(re^{i\theta}) \approx e^{i(d\theta+\alpha)} \text{ as } r \to \infty}$$

This is a model *quantized vortex.*

defects, continued

- Scaling leads to concentration: for example, if

$$E_\varepsilon(u) := \int_{\mathbb{R}^k} \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{2\varepsilon}(|u|^2 - 1)^2 \, dx, \qquad u \in H^1_{loc}(\mathbb{R}^k; \mathbb{R}^k)$$

  then corresponding defects are

$$u_\varepsilon(x) = u_1(\frac{x}{\varepsilon}).$$

- more generally/vaguely, a *topological defect* may refer to a structure that locally "looks like" a stationary defect.

Energy and evolution

The problems we study will have the form

$$\left.\begin{array}{l} u_t \\ u_{tt} \\ iu_t \end{array}\right\} + \nabla E_\varepsilon(u) = 0$$

for some E_ε on some Hilbert space, with $E_\varepsilon \to$ some limit E_0 as $\varepsilon \to 0$.

General question: relationship between limits of E_ε and of evolution equations.

Remark

At least formally,

- parabolic $\Rightarrow$ energy decreases: $\frac{d}{dt} E_\varepsilon(u_\varepsilon) = -|\partial_t u_\varepsilon|^2$.
- hyperbolic $\Rightarrow$ energy is conserved: $\frac{d}{dt}[\frac{1}{2}|\partial_t u_\varepsilon|^2 + E_\varepsilon(u_\varepsilon)] = 0$.
- Schrödinger $\Rightarrow$ (a different) energy is conserved: $\frac{d}{dt} E_\varepsilon(u_\varepsilon) = 0$.

The simplest questions of this sort can be solved by hand:

Exercise

Assume that $F_\varepsilon : \mathbb{R}^2 \cong \mathbb{C} \to \mathbb{R}$ and that $F_\varepsilon \to F$ in some topology as $\varepsilon \to 0$.
For which topologies is it true that solutions of the ODEs

$$\begin{aligned} \dot{x}_\varepsilon &= -\nabla F_\varepsilon(x_\varepsilon), \qquad && x_\varepsilon(0) = x_0 \\ \ddot{x}_\varepsilon &= -\nabla F_\varepsilon(x_\varepsilon), \qquad && x_\varepsilon(0) = x_0, \dot{x}_\varepsilon(0) = v_0 \\ i\dot{x}_\varepsilon &= -\nabla F_\varepsilon(x_\varepsilon), \qquad && x_\varepsilon(0) = x_0 \end{aligned}$$

converge, as $\varepsilon \to 0$, to solutions of the $\varepsilon = 0$ systems?

More precisely: find some k, α such that if $F_\varepsilon \to F$ in $C^{k,\alpha}$, and if $x_\varepsilon(\cdot)$ and $x(\cdot)$ solve any of the above equations[a] (take your pick) with the same initial data, then $x_\varepsilon(t) \to x(t)$ locally uniformly for $t > 0$.

[a] with F_ε replaced by F for $x(\cdot)$

In our setting, Γ-convergence is the relevant notion of convergence. To get some insight, we start with:

Definition (Γ-convergence on $\mathbb{R}^n$)

Assume that F_ε and F are functions on $\mathbb{R}^n$.
We say that $F_\varepsilon \xrightarrow{\Gamma} F$ if

1. For any sequence $(x_\varepsilon) \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, if $x_\varepsilon \to x$ then $\liminf F_\varepsilon(x_\varepsilon) \geq F(x)$
2. For every x, there exists $x_\varepsilon \to x$ such that $\limsup F_\varepsilon(x_\varepsilon) \leq F(x)$.

As the following examples suggest, Γ-convergence is too weak to allow any conclusions about dynamics.

Exercise

Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, $\phi : \mathbb{R}^n \to \mathbb{R}$ is measurable and $\mathbb{Z}^n$-periodic, with $\inf \phi = 0$. Then for any $p > 0$,

$$F_\varepsilon(x) := f(x) + \varepsilon^{-p}\phi(\frac{x}{\varepsilon}) \xrightarrow{\Gamma} f.$$

Exercise

Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is positive. Let $\{x_i\}$ be a countable dense subset of $\mathbb{R}^n$, and define

$$F_\varepsilon(x) := \begin{cases} 0 & \text{if } x \in \cup_{i=1}^\infty B(x_i, 2^{-i}\varepsilon) \\ f(x) & \text{if not} \end{cases}$$

Then

$$F_\varepsilon \to f > 0 \ a.e. \text{ and in } L^1_{loc}, \qquad \text{but} \quad F_\varepsilon(x) \xrightarrow{\Gamma} 0.$$

Nonetheless, for us, Γ-convergence will provide both inspiration and technical tools.

Plan for lectures

- codimension 1 defects: interfaces
  - review of relevant elliptic/Γ-convergence results
  - a sample problem – interfaces in NLW
- codimension 2 defects: quantized vortices
  - review of relevant Γ-convergence results
  - a sample problem – vortices in NLS

large open problems – not to be discussed

- Abstract framework for Γ-convergence and Hamiltonian systems.
  For gradient flows, Sandier-Serfaty '04, Serfaty '11
- Global-in-time results for Hamiltonian systems?
  - *e.g.*, given a periodic solution of a limiting Hamiltonian system, $\exists$? "nearby" periodic solutions of approximating Hamiltonian systems?

Part 1a: interfaces – background

We first consider interface motion in $u : (0, T) \times \mathbb{R}^n \to \mathbb{R}$ solving

$$u_{tt} - \Delta u + \frac{2}{\varepsilon^2}(u^2 - 1)u = 0.$$

We start by recalling the elliptic problem:

$$\boxed{-\Delta u + \frac{2}{\varepsilon^2}(u^2 - 1)u = 0}$$

in a bounded, open $\Omega \subset \mathbb{R}^n$, with boundary conditions on $\partial\Omega$.

This is the Euler-Lagrange equation for the Allen-Cahn energy

$$\boxed{E_\varepsilon(u) := \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{2\varepsilon}(u^2 - 1)^2 \, dx} \qquad u \in H^1_{loc}(\Omega; \mathbb{R}).$$

One can get insight into the equation

$$-\varepsilon \Delta u_\varepsilon + \frac{2}{\varepsilon}(u_\varepsilon^2 - 1)u_\varepsilon = 0$$

by the following formal argument:

- Seek approximate solutions of the form $\boxed{u_\varepsilon = q(\frac{d}{\varepsilon})}$ for $q : \mathbb{R} \to \mathbb{R}$ and $d : \Omega \to \mathbb{R}$ to be determined. Compute

$$\begin{aligned} -\varepsilon \Delta u_\varepsilon + &\frac{2}{\varepsilon}(u_\varepsilon^2 - 1)u_\varepsilon \\ &= \frac{1}{\varepsilon}\left[-q''(\frac{d}{\varepsilon})|\nabla d|^2 + 2(q^2(\frac{d}{\varepsilon}) - 1)q(\frac{d}{\varepsilon})\right] - q'(\frac{d}{\varepsilon})\Delta d. \end{aligned}$$

- This yields good aproximate solution if

$$\boxed{-q'' + 2(q^2 - 1)q = 0, \qquad |\nabla d|^2 = 1}$$

and

$$\boxed{-\Delta d \approx 0} \qquad \text{where } q'(\tfrac{d}{\varepsilon}) \text{ non-negligible.}$$

We thus consider

$$\begin{aligned} -q'' + 2(q^2 - 1)q &= 0 \\ |\nabla d|^2 &= 1 \\ -\Delta d &= 0\,. \end{aligned}$$

First, $\boxed{-q'' + 2(q^2 - 1)q = 0}$.

- Integrate to find
$$q'^2 = (1 - q^2)^2.$$
For boundary condition $q = \pm 1$ at $\pm\infty$ choose $q' = (1 - q^2)$.
- explicit solution: $q(s) = \tanh(s)$ (or translates thereof).
- optimality property: if $\tilde{q}(\pm\infty) = \pm 1$ then

$$\boxed{\frac{4}{3} = \frac{1}{2}\int_{\mathbb{R}} q'^2 + (q^2 - 1)^2 \leq \frac{1}{2}\int_{\mathbb{R}} \tilde{q}'^2 + (\tilde{q}^2 - 1)^2}$$

- This optimality property underlies many variational arguments.

Next, $\boxed{|\nabla d|^2 = 1}$.

- A class of solutions: say $O \subset \mathbb{R}^n$ with ∂O smooth. Define

$$d_O(x) := \begin{cases} \text{dist}(x, \partial O) & \text{if } x \in O \\ -\text{dist}(x, \partial O) & \text{if } x \in\not\in O. \end{cases}$$

  Then d_O smooth near ∂O, with $|\nabla d_O|^2 = 1$.

- May be described locally as follows:
  - Say $\psi : U \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$ parametrizes part of ∂O.
  - For $y' \in U$, let $\nu(y')$ = unit normal at $\psi(y')$ (inward, say). So

$$\boxed{(\nu(y'), \nu(y'))_e = 1}, \qquad \boxed{(\nu(y'), \partial_{y_k}\psi(y'))_e = 0 \text{ for } k = 1, \ldots, n-1.}$$

    Here $(\cdot, \cdot)_e$ denotes the Euclidean inner product.
  - For $y = (y', y_n) \in U \times (-r, r)$, define $\Psi(y) := \psi(y') + y_n \nu(y')$.
  - Let $\Phi = \Psi^{-1}$.

Lemma

Let $\zeta(y) := y_n$. Then $d := \zeta \circ \Phi$ solves $|\nabla d|^2 = 1$, with $d = 0$ on ∂O.

proof of the Lemma.

Compute

$$|\nabla d|^2 = \delta^{ij}\, \partial_{y_k}\zeta(\Phi)\, \partial_{x_i}\Phi^k\, \partial_{y_\ell}\, \zeta(\Phi)\, \partial_{x_j}\Phi^\ell$$

At $y = \Phi(x)$, define $g^{kl}(y) = \delta^{ij}\, \partial_i\Phi^k(x)\, \partial_j\Phi^\ell(x)$, and rewrite (suppressing arguments) as

$$|\nabla d|^2 = g^{kl}\, \partial_k\zeta\, \partial_\ell\zeta$$

Since $\nabla_y\zeta = (0, \ldots, 0, 1)$, this equals 1 by the exercise below. □

Exercise

Let $g_{k\ell}(y) := (\partial_k\Psi, \partial_\ell\Psi)_e$. Then $(g^{k\ell}(y)) = (g_{k\ell}(y))^{-1}$, and

$$(g_{k\ell}) = \begin{pmatrix} (\partial_{y_i}\psi, \partial_{y_j}\psi)_e & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}, \qquad \text{in block form, with } i, j = 1\ldots, n-1.$$

Finally, $\boxed{-\Delta d = 0}$ **when $d = 0$.**

- This is a geometric condition on $\{d = 0\}$.
- Let $n(x) := \nabla d(x)$. Then $n(\cdot)$ is field of unit normals to level sets of d.
- $$-\Delta d = -\partial_{x_i}(\delta^{ij} n_j) = -\text{div}\, n \overset{fact}{=} \text{mean curvature of level set}$$

---

conclusions: We expect solutions u_ε of the form

$$u_\varepsilon(x) \approx q(\frac{d(x)}{\varepsilon}),$$

where

- q solves $q' = 1 - q^2$, $q(0) = 0$, and $q(\pm\infty) = \pm 1$.
- $d(\cdot)$ is the signed distance to a minimal surface (*i.e.* a surface with mean curvature $\equiv 0$.)

Some rigorous results follow.....

Theorem (Modica-Mortola '77, Modica '87,Sternberg '88)

1. (compactness) *If* $(u_\varepsilon)_{\varepsilon\in(0,1]}$ *is a sequence in* $H^1(\Omega)$ *such that*

$$E_\varepsilon(u_\varepsilon) \le C$$

then there is a subsequence that converges in L^1 *as* $\varepsilon \to 0$ *to a limit* $u \in BV(\Omega; \{\pm 1\})$.

2. (lower bound) *If* $(u_\varepsilon) \subset H^1(\Omega)$ *and* $u_\varepsilon \stackrel{L^1}{\to} u$*, then*

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \ge E_0(u) := \begin{cases} \frac{2}{3}|Du|(\Omega) & \text{if } u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \text{if not} \end{cases}$$

3. (upper bound) *For any* $u \in L^1(\Omega)$ *there exists a sequence* $(u_\varepsilon) \subset H^1(\Omega; \mathbb{R})$ *such that*

$$u_\varepsilon \stackrel{L^1}{\to} u \quad \text{and } \limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \le E_0(u).$$

- Informally,

$$\boxed{E_\varepsilon(\cdot) \xrightarrow{\Gamma} \text{“ interfacial area functional ”}}$$

- As a **corollary**: if (u_ε) is a sequence of minimizers of E_ε (for suitable boundary data....) then

$u_\varepsilon \to u \in BV(\Omega; \{\pm 1\})$ after passing to a subsequence if necessary, and the set $\{x \in \Omega : u(x) = 1\}$ has minimal perimeter in Ω.

- Regularity results imply that if $\dim(\Omega) \leq 7$, then in fact

$\partial\{x \in \Omega : u(x) = 1\}$ is a smooth minimal surface in Ω.

further (PDE) results: (*many* references omitted here.....)

- similar for nonminimizing solutions of

$$-\varepsilon\Delta u_\varepsilon + \frac{2}{\varepsilon}(u_\varepsilon^2 - 1)u_\varepsilon = 0$$

  (assuming natural energy bounds.)

- solutions of

$$-\varepsilon\Delta u_\varepsilon + \frac{2}{\varepsilon}(u_\varepsilon^2 - 1)u_\varepsilon = \kappa$$

  are related in a similar way to surfaces of Constant Mean Curvature.

- in addition,

$u_\varepsilon(x) \approx q(\frac{d(x)}{\varepsilon})$, where $d(\cdot)$ is signed distance from interface,

so that $d(\cdot)$ satisfies $|\nabla d|^2 = 1, \quad d = 0 \text{ on } \Gamma.$

Γ-convergence Theorem: elements of proof

- Basic estimate: since $(1-u^2)u' \le \varepsilon u'^2 + \varepsilon^{-1}(1-u^2)^2$,

$$\int_\Omega |\nabla H(u_\varepsilon)| \le E_\varepsilon(u_\varepsilon), \qquad \text{for } H(u) := u - \frac{1}{3}u^3.$$

$$\int_\Omega |H(u_\varepsilon)| \le C \int_\Omega (1 + (u_\varepsilon^2 - 1)^2).$$

- If $E_\varepsilon(u_\varepsilon) \le C$, standard compactness result $\Rightarrow$

$$H(u_\varepsilon) \to H \text{ in } L^1(\Omega).$$

(after passing to subsequence)

Exercise

Then $u_\varepsilon \to$ some limit u in $L^1(\Omega)$. Moreover,

- $H(u) = H$. That is, $H(\lim u_\varepsilon) = \lim H(u_\varepsilon)$.
- Since $u_\varepsilon^2 \to 1$ in $L^2(\Omega)$, necessarily $|u| = 1$ *a.e.*

proof cont'd

Definition

$u \in BV(\Omega)$ if $u \in L^1(\Omega)$ and

$$|Du|(\Omega) := \sup \left\{ \int_\Omega u \, \nabla \cdot \phi dx : \phi \in C_c^\infty(\Omega; \mathbb{R}^n), \|\phi\|_\infty \le 1 \right\} < \infty.$$

- Since $|u| = 1$ a.e., $H(u) = H = \frac{2}{3}u$. Thus if $\phi \in C_c^\infty(\Omega; \mathbb{R}^n)$ and $\|\phi\|_\infty \le 1$

$$\frac{2}{3}\int_\Omega u \, \nabla \cdot \phi dx = \int_\Omega H \, \nabla \cdot \phi dx = \lim \int_\Omega H(u_\varepsilon) \, \nabla \cdot \phi dx$$
$$= -\lim \int_\Omega \nabla H(u_\varepsilon) \cdot \phi dx \le \|\nabla H(u_\varepsilon)\|_{L^1} \le \liminf E_\varepsilon(u_\varepsilon)$$

- sup over ϕ to complete compactness and prove lower bound.

proof cont'd: upper bound

- first consider 1 dimension: if $q_\varepsilon(x) \to \pm 1$ as $x \to \pm\infty$, then

$$\int \frac{\varepsilon}{2} q_\varepsilon'^2 + \frac{1}{2\varepsilon}(1 - q_\varepsilon^2)^2 \geq \int (1 - q_\varepsilon^2) q_\varepsilon' = \int H(q_\varepsilon)'$$

  with equality if $\boxed{q_\varepsilon' = \frac{1}{\varepsilon}(1 - q_\varepsilon^2)}$.

- Above ODE thus characterizes optimal profile.
- If $u = \mathbf{1}_O - \mathbf{1}_{\Omega \setminus O}$ for ∂O smooth, define signed distance function

$$d(x) := \begin{cases} \mathrm{dist}(x, \partial O) & \text{if } x \in O \\ -\mathrm{dist}(x, \partial O) & \text{if } x \in \Omega \setminus O. \end{cases}$$

  Good trial functions are:

$$u_\varepsilon(x) := q_\varepsilon(d(x)) = q(\frac{d(x)}{\varepsilon}), \qquad \text{where } q := q_1$$

- general case follows by approximation argument

Exercise

Show that if $\Omega = \mathbb{R}^n$ and $u = \mathbf{1}_O - \mathbf{1}_{\mathbb{R}^n \setminus O}$, where O is a bounded, open set with smooth boundary, then for u_ε defined above,

$$E_\varepsilon(u_\varepsilon) \to \frac{2}{3}|Du|(\mathbb{R}^n) = \frac{4}{3}\mathcal{H}^{n-1}(\partial O).$$

One way to do this is, as above,

- parametrize pieces of ∂O by $\psi : U \to \partial O$
- change of variables via $\Psi = \psi(y') + y_n \nu(y')$ and $\Phi = \Psi^{-1}$.
- Define (g_{ij}) and (g^{ij}) as above, and let $g := \det(g_{ij})$.

Then setting $v_\varepsilon = u_\varepsilon \circ \Psi = q(\frac{y_n}{\varepsilon})$,

$$\int_{\mathrm{Image}(\Psi)} e_\varepsilon(u_\varepsilon) = \int_U \int_{-r}^{r} [\frac{\varepsilon}{2} g^{ij} \partial_{y_i} v_\varepsilon \, \partial_{y_j} v_\varepsilon + \frac{1}{2\varepsilon}(1 - v_\varepsilon^2)^2] \sqrt{g} dy_n \, dy'$$

$$= \frac{1}{2\varepsilon} \int_U \int_{-r}^{r} [q'^2(\frac{y_n}{\varepsilon}) + (q(\frac{y_n}{\varepsilon})^2 - 1)^2] \sqrt{g} dy_n \, dy'.$$

Also, useful: $|q(s) - \mathrm{sign}(s)| \le Ce^{-cs}$ and $|q'(s)| \le Ce^{-cs}$.

1d estimates, for future use

Lemma

If $u \in H^1(I)$ for $I \subset \mathbb{R}$, then for every interval $J \subset I$ of length ε,

$$\sup_J (\min_{\pm} |1 \pm u|^2) \le C \int_J \frac{\varepsilon}{2} u'^2 + \frac{1}{2\varepsilon}(u^2 - 1)^2$$

Lemma

For $\rho > 0$, there exists $c_1(\rho)$ s.t. if $\int_{-\rho}^{\rho} |v(s) - \mathrm{sign}(s)|^2 \, |s| \, ds \le c_1$ then for any $\bar\rho \ge \rho$,

$$\int_{-\bar\rho}^{\bar\rho} (\sqrt{\varepsilon} v' - \frac{1}{\sqrt{\varepsilon}}(1 - v^2))^2 \le C \left[\int_{-\bar\rho}^{\bar\rho} \frac{\varepsilon}{2} v'^2 + \frac{1}{2\varepsilon}(1 - v^2)^2 \, ds - \frac{4}{3} \right] + Ce^{-c/\varepsilon}$$

and in particular $$\boxed{\int_{-\bar\rho}^{\bar\rho} \frac{\varepsilon}{2} v'^2 + \frac{1}{2\varepsilon}(1 - v^2)^2 \, ds \ge \frac{4}{3} - Ce^{-c/\varepsilon}}.$$

Part 1b: interfaces in a semilinear wave equation

We now consider interface motion in $u : (0, T) \times \mathbb{R}^n \to \mathbb{R}$ solving

$$u_{tt} - \Delta u + \frac{2}{\varepsilon^2}(u^2 - 1)u = 0.$$

This is the Euler-Lagrange equation for the action functional

$$\boxed{A_\varepsilon(u) := \int_\Omega \frac{\varepsilon}{2}\eta^{\alpha\beta}\,\partial_{x_\alpha} u\,\partial_{x_\beta} u + \frac{1}{2\varepsilon}(u^2 - 1)^2\, dx} \qquad u \in H^1_{loc}(\Omega; \mathbb{R}).$$

where $(\eta^{\alpha\beta}) = \mathrm{diag}(-1, 1, \ldots, 1)$ is the Minkowski metric.
Here and below we identify $t \approx x_0$

Formal asymptotics are almost exactly as in the elliptic case (with some changes of sign):

heuristic argument

- Seek approximate solutions of the form $\boxed{u_\varepsilon = q(\frac{d}{\varepsilon})}$ for $q : \mathbb{R} \to \mathbb{R}$ and $d : \Omega \to \mathbb{R}$ to be determined. Compute

$$\varepsilon \eta^{\alpha\beta} \partial_{x_\alpha x_\beta} u_\varepsilon + \frac{2}{\varepsilon}(u_\varepsilon^2 - 1) u_\varepsilon = \frac{1}{\varepsilon}\left[-q''(\frac{d}{\varepsilon}) \eta^{\alpha\beta} \partial_{x_\alpha} d\, \partial_{x_\beta} d + 2(q^2(\frac{d}{\varepsilon}) - 1) q(\frac{d}{\varepsilon}) \right] + q'(\frac{d}{\varepsilon}) \eta^{\alpha\beta} \partial_{x_\alpha x_\beta} d.$$

- This yields good aproximate solution if (writing $D := (\partial_t, \nabla)$)

$$\boxed{-q'' + 2(q^2 - 1)q = 0,}$$

$$\boxed{\eta^{\alpha\beta} \partial_{x_\alpha} d\, \partial_{x_\beta} d = -(\partial_t d)^2 + |\nabla d|^2 = (Dd, Dd)_m = 1}$$

$$\boxed{\eta^{\alpha\beta} \partial_{x^\alpha x^\beta} d = (\partial_{tt} - \Delta) d \approx 0}$$

where $q'(\frac{d}{\varepsilon})$ non-negligible.

We thus consider

$$-q'' + 2(q^2 - 1)q = 0$$
$$(Dd, Dd)_m = 1$$
$$(\partial_{tt} - \Delta)d = 0$$

First, $\boxed{-q'' + 2(q^2 - 1)q = 0}$. This is exactly as before:

- Integrate to find
$$q'^2 = (1 - q^2)^2.$$
For boundary condition $q = \pm 1$ at $\pm\infty$ choose $q' = (1 - q^2)$.
- explicit solution: $q(s) = \tanh(s)$ (or translates thereof).
- optimality property: if $\tilde{q}(\pm\infty) = \pm 1$ then

$$\boxed{\frac{4}{3} = \frac{1}{2}\int_{\mathbb{R}} q'^2 + (q^2 - 1)^2 \leq \frac{1}{2}\int_{\mathbb{R}} \tilde{q}'^2 + (\tilde{q}^2 - 1)^2}$$

- This optimality property underlies many variational arguments.

Next, $\boxed{\eta^{\alpha\beta}\partial_{x_\alpha} d\, \partial_{x_\beta} d = -(\partial_t d)^2 + |\nabla d|^2 = (Dd, Dd)_m = 1}$.

Here $(\cdot,\cdot)_m$ denotes the Minkowski inner product.

May be constructed locally as follows:

- Say $\psi : U \subset \mathbb{R}^n \to \mathbb{R}^{1+n}$ parametrizes part of $\Gamma \subset (0,T)\times\mathbb{R}^n$.
- For $y' \in U$, let $\nu(y')$ = unit normal at $\psi(y')$. So

$$\boxed{(\nu(y'), \nu(y'))_m = 1,} \qquad \boxed{(\nu(y'), \partial_{y_k}\psi(y'))_m = 0 \text{ for } k = 0, \dots, n-1.}$$

- For $y = (y', y_n) \in U \times (-r, r)$, define $\Psi(y) := \psi(y') + y_n\,\nu(y')$.
- Let $\Phi = \Psi^{-1}$.

Lemma

Let $\zeta(y) := y_n$. Then $d := \zeta\circ\Phi$ solves $|\nabla d|^2 = 1$, with $d = 0$ on Γ.

We will call the $d(\cdot)$ the *signed Minkowski distance* to Γ.

Remark 1
Note, in Minkiwski space, the vectors

$$v_0 := (\cosh\theta, \sinh\theta, 0, \dots, 0)$$

$$v_1 := (\sinh\theta, \cosh\theta, 0, \dots, 0)$$

satisfy (for any $\theta \in \mathbb{R}$)

$$(v_0, v_0)_m = -1,$$
$$(v_1, v_1)_m = 1,$$
$$(v_0, v_1)_m = 0.$$

Remark 2
A hypersurface Γ is said to be *timelike* if and only if at every point it has a unit normal ν such that $(\nu, \nu)_m = 1$.

proof of the Lemma.

Compute

$$\eta^{\alpha\beta}\partial_{x_\alpha} d\, \partial_{x_\beta} d = \eta^{\alpha\beta}\, \partial_{y_\mu}\zeta(\Phi)\, \partial_{x_\alpha}\Phi^\mu\, \partial_{y_\nu}\, \zeta(\Phi)\, \partial_{x_\beta}\Phi^\nu$$

At $y = \Phi(x)$, define $g^{\mu\nu}(y) = \eta^{\alpha\beta}\partial_{x_\alpha}\Phi^\mu(x)\, \partial_{x_\beta}\Phi^\nu(x)$ and rewrite (suppressing arguments) as

$$(Dd, Dd)_m = g^{\mu\nu}\, \partial_\mu\zeta\, \partial_\nu\zeta$$

Since $D\zeta = (0, \ldots, 0, 1)$, this equals 1 by the exercise below. □

Exercise

Let $g_{\mu\nu}(y) := (\partial_\mu\Psi, \partial_\nu\Psi)_m$. Then $(g^{\mu\nu}(y)) = (g_{\mu\nu}(y))^{-1}$, and

$$(g_{\mu\nu}) = \begin{pmatrix} (\partial_{y_a}\psi, \partial_{y_b}\psi)_m & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}, \qquad \text{in block form, with } a, b = 0\ldots, n-1.$$

Finally, $\boxed{(\partial_{tt} - \Delta)d = 0}$ **when** $d = 0$**.**

- This is a geometric condition on $\{d = 0\}$.
- Let $n^\alpha(x) := \eta^{\alpha\beta}\partial_\beta d(x)$. Then $n(\cdot)$ is field of (Minkowski) unit normals to level sets of d.
- $$(\partial_{tt} - \Delta)d = -\partial_{x_\alpha}(n^\alpha) = -\text{div}\, n$$
  $$\overset{fact}{=} \text{Minkowski mean curvature of level set}$$

---

conclusions: We expect solutions u_ε of the form

$$u_\varepsilon(x) \approx q(\frac{d(x)}{\varepsilon}),$$

where

- q solves $q' = 1 - q^2$, $q(0) = 0$, and $q(\pm\infty) = \pm 1$.
- $d(\cdot)$ is the signed Minkowski distance to a timelike hypersurface with Minkowski mean curvature $\equiv 0$.)

Some rigorous results follow.....

Note: Consider Γ that can be represented as a graph:

$$\Gamma = \{(t, x, h(t,x)) : (t,x) \in \mathbb{R} \times \mathbb{R}^{n-1}\}$$

for some $h : \mathbb{R}^n \to \mathbb{R}$. One choice of unit normal is

$$n = (n^\alpha) = \frac{(-\partial_t h, \nabla h, -1)}{\sqrt{1 + |\nabla h|^2 - (\partial_t h)^2}}$$

(Here assuming $(h_t)^2 < 1 + |\nabla h|^2$, which says that Γ is timelike.) Hence

$$\begin{aligned} H_{mink}(\Gamma) &= -\operatorname{div} n \\ &= \partial_t\Big(\frac{\partial_t h}{\sqrt{1 + |\nabla h|^2 - (\partial_t h)^2}}\Big) - \nabla \cdot \Big(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2 - (\partial_t h)^2}}\Big). \end{aligned}$$

From this one can see that

$$H_{mink}(\Gamma) = \text{Minkowskian mean curvature of } \Gamma \ = \kappa \in \mathbb{R} \qquad (1)$$

is a quasilinear wave equation. Local existence of smooth solutions is known.

Other facts about (timelike) Minkowskian mean curvature:

- In general,

$$H_{mink} = (1 - v^2)^{-1/2}(H_{euc} - (1 - v^2)^{-1} a),$$

where $v = \textit{velocity}$, $a = \textit{acceleration}$.

- Thus, heuristically,

$$H_{mink} = \text{(relativistic)} \Big[\text{(euclidean) mean curvature} - \text{acceleration}\Big]$$

- If $\Gamma = (t, x, h(t, x)) : (t, x) \in O \subset \mathbb{R}^n\}$ is timelike and O has finite measure, then the Minkowskian area of Γ

$$\int_O \sqrt{1 + |\nabla h|^2 - (\partial_t h)^2}\, dx\, dt$$

The equation $H_{mink} = 0$ is the Euler-Lagrange equation associated to the Minkowskian area functional.

Theorem (J '11, Galvão-Sousa and J., '15)

Let Γ be a smooth, timelike hypersurface in $(T_*, T^*) \times \mathbb{R}^n$, bounding a set $\mathcal{O}$, and such that $H_{mink}(\Gamma) = \kappa \in \mathbb{R}$.

Then there exists a sequence of solutions (u_ε) of the wave equation

$$\varepsilon(\partial_{tt} u_\varepsilon - \Delta u_\varepsilon) + \frac{1}{\varepsilon}(u^2 - 1)(2u - \varepsilon\kappa) = 0$$

such that

$$u_\varepsilon \to u := \begin{cases} 1 & \text{in } \mathcal{O} \\ -1 & \text{in } \mathcal{O}^c \end{cases} \qquad \text{in } L^2_{loc}((T_*, T^*) \times \mathbb{R}^n)$$

- In fact we prove more, including energy concentration around Γ, estimates of rate of convergence *etc.*

Remark

Balance laws for PDEs and geometric problems have similar structure:

- a solution $u_\varepsilon \in H^1(\Omega;\mathbb{R})$ of the elliptic equation satisfies

$$\int \nabla X : \left(I - \frac{\varepsilon \nabla u_\varepsilon \otimes \nabla u_\varepsilon}{e_\varepsilon(u_\varepsilon)}\right) e_\varepsilon(u_\varepsilon)\, dx \;=\; 0 \qquad \text{for all } X \in C_c^\infty(\Omega;\mathbb{R}^n).$$

- A smooth minimal surface $\Gamma \subset \Omega$ satisfies an identity we can write as

$$\int_\Gamma \nabla X : (I - P^\perp)\, d\mathcal{H}^{n-1} \;=\; 0 \qquad \text{for all } X \in C_c^\infty(\Omega;\mathbb{R}^n).$$

The latter identity is the basis for the definition of a *stationary varifold*

Remark cont'd

- a smooth solution $u_\varepsilon : (0, T) \times \mathbb{R}^n \to \mathbb{C}$ of the parabolic equation satisfies

$$\frac{d}{dt} \int \phi \, e_\varepsilon(u_\varepsilon) dx = - \int \phi \, \varepsilon |\partial_t u_\varepsilon|^2 \, dx - \int \nabla^2 \phi : (I - \frac{\varepsilon \nabla u_\varepsilon \otimes \nabla u_\varepsilon}{e_\varepsilon(u_\varepsilon)}) \, e_\varepsilon(u_\varepsilon) \, dx$$

  for all $\phi \in C_c^\infty(\mathbb{R}^n)$.
- if $(\Gamma_t)_{t \in (0,\infty)}$ is a smooth family of codimension 1 submanifolds evolving by mean curvature, then

$$\frac{d}{dt} \int_{\Gamma_t} \phi \, d\mathcal{H}^{n-1} = - \int_{\Gamma_t} \left[\phi |H|^2 + \nabla^2 \phi : (I - P^\perp) \right] d\mathcal{H}^{n-1} dx$$

  for all $\phi \in C_c^\infty(\mathbb{R}^n)$, where H denotes the mean curvature vector along Γ_t.

The latter identity is the basis for the definition of a *Brakke flow*

Remark cont'd

- a smooth solution $u_\varepsilon : (0,T)\times\mathbb{R}^n \to \mathbb{C}$ of the wave equation satisfies

$$\int DX : (\eta - \frac{\varepsilon(\eta\, Du_\varepsilon)\otimes(\eta\, Du_\varepsilon)}{\ell_\varepsilon(u_\varepsilon)})\ \ell_\varepsilon(u_\varepsilon)\, dx\, dt \ = \ 0$$

  for $X \in C_c^\infty(\mathbb{R}^{1+N};\mathbb{R}^{1+n})$, where $Du = (\partial_t u, \nabla u)$,
  $\ell_\varepsilon(u_\varepsilon) =$lagrangian.
- If Γ is a smooth timelike submanifold of $\mathbb{R}^{1+n}$ with vanishing mean curvature, then

$$\int_\Gamma DX : (\eta - P^\perp_{mink})\, d\lambda \ = \ 0 \qquad \text{for all } X \in C_c^\infty(\mathbb{R}^{1+n};\mathbb{R}^{1+n})$$

  where λ denotes the Minkoskian area measure, and $P^\perp_{mink}$ denotes Minkowski orthogonal projection onto $(T_{(t,x)}\Gamma)^{\perp,mink}$.

The latter is the basis for a definition of *stationary Minkowskian varifolds* (Bellettini, Novaga, Orlandi 2010)

Starting point: Change variables via the diffeomorphism $\Psi(y) = \psi(y') + y_n\nu(y')$, for $y' = (y_0, \ldots, y_{n-1})$. Define

$$g_{\alpha\beta} := (\partial_{y_\alpha}, \partial_{y_\beta})_m, \qquad (g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}, \qquad g = \det(g_{\alpha\beta}).$$

Then $v = u \circ \Psi$ satisfies

$$-\frac{\varepsilon}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta v) + \frac{1}{\varepsilon}(v^2 - 1)(2v - \varepsilon\kappa) = 0.$$

Rewrite

$$-\varepsilon\partial_\alpha(g^{\alpha\beta}\partial_\beta v) - \varepsilon B^\alpha\partial_\alpha v + \frac{1}{\varepsilon}(v^2 - 1)(2v - \varepsilon\kappa) = 0,$$

for

$$B^\alpha := \frac{1}{\sqrt{-g}}g^{\alpha\beta}\partial_\beta\sqrt{-g}.$$

Fact

$$H_{mink}(\Gamma) = \kappa \quad \Longleftrightarrow \quad B^n(y', y_n) = \kappa + O(|y_n|).$$

Core of the proof: weighted energy estimates in normal coordinates.

For simplicity, assume that Γ is parametrized by $\psi : (T_*, T^*) \times \mathbb{T}^{n-1} \to \mathbb{R}^{1+n}$

Define $\rho(s) = \rho_0 - c_* s$ and

$$\zeta_1(s) := \int_{\mathbb{T}^{n-1}} \left[\int_{-\rho(s)}^{\rho(s)} (1 + y_n^2) \Big(\tfrac{\varepsilon}{2} a^{\alpha\beta} \partial_{y_\alpha} v \partial_{y_\beta} v + \tfrac{(1-v^2)^2}{2\varepsilon} \Big) dy_n - \frac{4}{3} \right] dy' \Bigg|_{y_0 = s}$$

$$\zeta_2(s) := \int_{\mathbb{T}^{n-1}} \int_{-\rho_0/2}^{\rho_0/2} |v - \mathrm{sign}(y_n)|^2 \, |y_n| \, dy_n dy' \Bigg|_{y_0 = s}$$

$$\zeta_3(s) := \int_{\mathbb{T}^{n-1}} \int_{-\rho(s)}^{\rho(s)} \varepsilon |D_\tau v|^2 + (y_n)^2 \Big(\varepsilon (\partial_n v)^2 + \frac{1}{2\varepsilon} (1 - v^2)^2 \Big) dy_n dy' \Bigg|_{y_0 = s}$$

---

main estimates:

$$\zeta_1' \le C(\zeta_1 + \zeta_2 + \zeta_3) + Ce^{-c/\varepsilon}$$

$$\zeta_2 \le 2\zeta_2(0) + C \int_0^s \zeta_3(s) \, ds$$

$$\zeta_3 \le \zeta_1 + C\zeta_2 + Ce^{-c/\varepsilon}.$$