Namba-like Forcings at Successors of Singular Cardinals

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Infinity, Computability, and Metamathematics:
Celebrating the 60th Birthdays of Peter
Koepke and Philip Welch
Hausdorff Center for Mathematics
Bonn, Germany
May 23-25, 2014

We begin by recalling that *Namba forcing* is a notion of forcing definable in any model of ZFC which changes the cofinality of $\aleph_2$ to ω without collapsing any cardinals other than $\aleph_2$. Assuming CH holds, Namba forcing doesn't add any new reals. Covering considerations tell us that if some "reasonable analogue" of Namba forcing were to exist at a cardinal $\kappa > \aleph_2$, then at the minimum, $0^\sharp$ must exist.

On March 22, 2013, Peter Koepke gave a lecture in the CUNY Logic Workshop titled "Namba-like singularizations of successor cardinals" in which he discussed joint work with Dominik Adolf concerning how to obtain Namba-like forcings at successors of regular cardinals, assuming certain large cardinal hypotheses. His lecture was very inspiring, and caused me to begin thinking about how to obtain similar results at successors of singular cardinals. This is the topic on which I will speak.

Assuming κ is a singular cardinal, we will be looking to define a (set) partial ordering $\mathbb{P}$ such that

1. κ remains a singular cardinal in $V^{\mathbb{P}}$ of the same cofinality as in V.

2. $\mathrm{cof}^{V^{\mathbb{P}}}((\kappa^+)^V) < \kappa$.

3. Cardinals above $(\kappa^+)^V$ are preserved.

Note that assuming the appropriate hypotheses, Woodin's stationary tower forcing $\mathbb{P}_{<\lambda}$ can be defined so as to have properties (1) and (2). However, the chain condition on $\mathbb{P}_{<\lambda}$ is large, and forcing with $\mathbb{P}_{<\lambda}$ will collapse many cardinals above κ^+.

If we are able to define a (set) partial ordering $\mathbb{P}$ with properties (1) and (2) above, then this already implies strong hypotheses. Specifically, we have the following.

Theorem 1 *Suppose that it is possible to define a set partial ordering $\mathbb{P}$ such that for some singular cardinal κ*

1. *κ remains a singular cardinal in $V^{\mathbb{P}}$.*

2. $\text{cof}^{V^{\mathbb{P}}}((\kappa^+)^V) < \kappa$.

Then there must be an inner model with a Woodin cardinal.

Proof: Let $\mathbb{P} \in V$ and κ be as in the hypotheses for Theorem 1. Assume that there is no inner model with a Woodin cardinal. Then it is possible to build K within V. Further, we know that K computes successors of singular cardinals correctly. This means that

$$(\kappa^+)^K = (\kappa^+)^V.$$

Consider now $V^{\mathbb{P}}$. By the absoluteness of K and its properties under set forcing, it is possible to build the same K within $V^{\mathbb{P}}$. Since by (1) κ remains a singular cardinal in $V^{\mathbb{P}}$, it is still the case that

$$(\kappa^+)^K = (\kappa^+)^{V^{\mathbb{P}}}.$$

However, since (2) implies that

$$(\kappa^+)^V < (\kappa^+)^{V^{\mathbb{P}}},$$

we have that

$$(\kappa^+)^K = (\kappa^+)^V < (\kappa^+)^{V^{\mathbb{P}}} = (\kappa^+)^K.$$

This contradiction completes the proof of Theorem 1.

$\square$

Thus, in order to define our desired partial orderings $\mathbb{P}$, we must use hypotheses at least as strong as the existence of a Woodin cardinal. In fact, we will use much stronger hypotheses. Specifically, we have several theorems, starting with the following.

Theorem 2 *Suppose that in V, $\kappa = \sup_{i<\omega} \kappa_i$, where each κ_i is κ^+ strongly compact. Then there is a Namba-like partial ordering $\mathbb{P}$ such that*

1. *Forcing with $\mathbb{P}$ adds no new bounded subsets of κ (so in particular, κ remains a cardinal in $V^{\mathbb{P}}$).*

2. $\mathrm{cof}^{V^{\mathbb{P}}}((\kappa^+)^V) = \omega$.

3. *$\mathbb{P}$ is κ^{++}-c.c. (so in particular, cardinals above $(\kappa^+)^V$ are preserved).*

Proof: The definition of $\mathbb{P}$ uses ideas of Dehornoy, Gitik, Henle, Magidor, Sargsyan, and others. Since each κ_i is κ^+ strongly compact, let $\langle \mathcal{U}_i \mid i < \omega \rangle$ be such that $\mathcal{U}_i$ is a κ_i-additive, uniform ultrafilter over κ^+.

$\mathbb{P}$ may now be defined as the set of all finite sequences of the form $\langle \alpha_1, \ldots, \alpha_n, f \rangle$ satisfying the following properties.

1. $\langle \alpha_1, \ldots, \alpha_n \rangle \in [\kappa^+]^{<\omega}$.

2. f is a function having domain $T_{\alpha_1,\ldots,\alpha_n} = \{\langle \beta_1, \ldots, \beta_m \rangle \in [\kappa^+]^{<\omega} \mid \langle \alpha_1, \ldots, \alpha_n \rangle$ is an initial segment of $\langle \beta_1, \ldots, \beta_m \rangle\}$ such that $f(\langle \beta_1, \ldots, \beta_m \rangle) \in \mathcal{U}_m$.

The ordering on $\mathbb{P}$ is given by $\langle \beta_1, \ldots, \beta_m, g \rangle$ extends $\langle \alpha_1, \ldots, \alpha_n, f \rangle$ iff the following criteria are met.

1. $\langle \alpha_1, \ldots, \alpha_n \rangle$ is an initial segment of $\langle \beta_1, \ldots, \beta_m \rangle$.

2. For $i = n + 1, \ldots, m$, $\beta_i \in f(\langle \alpha_1, \ldots, \alpha_n, \ldots, \beta_{i-1} \rangle)$.

3. For every $\vec{s} \in \text{dom}(g)$ (which must be a subset of dom(f)), $g(\vec{s}) \subseteq f(\vec{s})$.

The usual density argument shows that forcing with $\mathbb{P}$ adds a cofinal ω sequence to $(\kappa^+)^V$. It is possible to prove a Prikry lemma for $\mathbb{P}$, i.e., given $\langle \alpha_1, \ldots, \alpha_n, f \rangle \in \mathbb{P}$ and formula φ in the language of forcing with respect to $\mathbb{P}$, there is a condition $\langle \alpha_1, \ldots, \alpha_n, f' \rangle$ extending $\langle \alpha_1, \ldots, \alpha_n, f \rangle$ deciding φ. If we then let $m < \omega$ be arbitrary and assume that we are extending conditions with stems having length at least m, by the fact each $\mathcal{U}_i$ is κ_i-additive, the usual proof shows that forcing with $\mathbb{P}$ adds no new bounded subsets of κ_m. Thus, forcing with $\mathbb{P}$ adds no new bounded subsets of κ. Since any two conditions having the same stem are compatible, $\mathbb{P}$ is κ^{++}-c.c. This completes the proof sketch of Theorem 2.

$\square$

Theorem 2 shows that under the appropriate large cardinal hypotheses, a Namba-like forcing outright exists at successors of certain singular cardinals. If we are willing to consider relative consistency results, then it is possible to establish additional theorems. More specifically, we have the following.

Theorem 3 *Con(ZFC + There exists a cardinal κ which is κ^+ strongly compact) $\implies$ Con(ZFC + There is a singular cardinal κ having cofinality ω and a κ^{++}-c.c. Namba-like forcing $\mathbb{P}$ such that forcing with $\mathbb{P}$ adds no new* *bounded subsets of κ and changes the cofinality of $(\kappa^+)^V$ to ω).*

Proof: Suppose $V \models$ "κ is κ^+ strongly compact". Fix $\mathcal{U}$ a κ-additive, fine measure over $P_\kappa(\kappa^+)$. We may now define strongly compact Prikry forcing $\mathbb{Q}$ as the set of all finite sequences of the form $\langle p_1, \ldots, p_n, f \rangle$ satisfying the following properties.

1. Each $p_i \in P_\kappa(\kappa^+)$.

2. $p_1 \subseteq \cdots \subseteq p_n$.

3. f is a function having domain $T_{p_1,\ldots,p_n} = \{\langle q_1, \ldots, q_m \rangle \mid q_1 \subseteq \cdots \subseteq q_m$ and $\langle p_1, \ldots, p_n \rangle$ is an initial segment of $\langle q_1, \ldots, q_m \rangle\}$ such that $f(\langle q_1, \ldots, q_m \rangle) \in \mathcal{U}$.

The ordering on $\mathbb{P}$ is given by $\langle q_1, \ldots, q_m, g \rangle$ extends $\langle p_1, \ldots, p_n, f \rangle$ iff the following criteria are met.

1. $\langle p_1, \ldots, p_n \rangle$ is an initial segment of $\langle q_1, \ldots, q_m \rangle$.

2. For $i = n+1, \ldots, m$, $q_i \in f(\langle p_1, \ldots, p_n, \ldots, q_{i-1} \rangle)$.

3. For every $\vec{s} \in \text{dom}(g)$ (which must be a subset of $\text{dom}(f)$), $g(\vec{s}) \subseteq f(\vec{s})$.

Since any two conditions having the same stem are compatible, $\mathbb{Q}$ is κ^{++}-c.c. Forcing with $\mathbb{Q}$ adds no new bounded subsets of κ and generates an ω sequence $r = \langle p_i \mid i < \omega \rangle$ which changes the cofinality of both κ and $(\kappa^+)^V$ to ω and also collapses $(\kappa^+)^V$ to κ. And, if we let $r \restriction \kappa = \langle p_i \cap \kappa \mid i < \omega \rangle$, in $V[r \restriction \kappa]$, κ is a singular cardinal having cofinality ω. Further, $(\kappa^+)^{V[r \restriction \kappa]} = (\kappa^+)^V$, and since $V[r \restriction \kappa] \subseteq V[r]$, V, $V[r \restriction \kappa]$, and $V[r]$ all contain the same bounded subsets of κ.

Working now in $V[r \restriction \kappa]$, let $\mathbb{P} = \mathbb{Q}/(r \restriction \kappa)$, i.e., $\mathbb{P}$ is the quotient forcing of $\mathbb{Q}$ with respect to $r \restriction \kappa$. $\mathbb{P}$ is the desired Namba-like forcing over $V[r \restriction \kappa]$. This is since $V[r \restriction \kappa]$ and $(V[r \restriction \kappa])^{\mathbb{P}} = V^{\mathbb{Q}}$ contain the same bounded subsets of κ, $\text{cof}^{V^{\mathbb{Q}}}((\kappa^+)^{V[r\restriction\kappa]}) = \omega$, and $\mathbb{P} = \mathbb{Q}/(r \restriction \kappa)$ is κ^{++}-c.c. because $\mathbb{Q}$ is. This completes the proof sketch of Theorem 3.

$\square$

Using Magidor's work from his paper "On the Singular Cardinals Problem I" and the ideas of Theorem 3, it is possible to transfer the results of Theorem 3 down to $\aleph_\omega$ and $\aleph_{\omega_1}$. Specifically, we have the following.

Theorem 4 *Con(ZFC + There exists a cardinal κ which is κ^+ supercompact) $\Longrightarrow$ Con(ZFC + There is a Namba-like forcing $\mathbb{P}$ such that forcing with $\mathbb{P}$ adds no new bounded subsets of $\aleph_\omega$, changes the cofinality of $(\aleph_{\omega+1})^V$ to ω, and preserves cardinals above $(\aleph_{\omega+1})^V$).*

Theorem 5 *Con(ZFC + There exists a cardinal κ such that $2^{\kappa^+} = \kappa^{++}$ and κ is κ^{++} supercompact) $\Longrightarrow$ Con(ZFC + There is a Namba-like forcing $\mathbb{P}$ such that forcing with $\mathbb{P}$ changes the cofinality of $(\aleph_{\omega_1+1})^V$ to ω_1, preserves $\aleph_{\omega_1}$, and preserves cardinals above $(\aleph_{\omega_1+1})^V$).*

The proofs of Theorems 4 and 5 are slight modifications of the proof of Theorem 3. The proof of Theorem 5 uses supercompact Magidor forcing based on $P_\kappa(\kappa^+)$ interleaved with Lévy collapses to collapse the κ^{++} supercompact cardinal κ to $\aleph_{\omega_1}$ while simultaneously changing κ's cofinality to ω_1 as Magidor does in Singular Cardinals I. Therefore, the Namba-like forcing of Theorem 5 will both add bounded subsets of $\aleph_{\omega_1}$ and collapse cardinals below $\aleph_{\omega_1}$.

Theorems 2 − 5 raise a number of questions. In particular:

1. Are there other successors of singular cardinals at which one can outright prove that a Namba-like forcing exists, assuming the appropriate large cardinal hypotheses?

2. Is it possible to weaken the large cardinal assumptions used to prove Theorems 2 – 5? What are the optimal hypotheses?

3. Is it possible to prove versions of Theorems 2 – 5 in which the cofinality of the successor of the singular cardinal is changed to a cofinality different from the singular cardinal's? This can be done with the stationary tower forcing.

4. Is it possible to prove a version of Theorem 5 in which no bounded subsets of $\aleph_{\omega_1}$ are added by the Namba-like forcing? More weakly, is it possible to prove a version of Theorem 5 in which no cardinals below $\aleph_{\omega_1}$ are collapsed? This can be done with the stationary tower forcing.

5. What applications (if any) are there for the Namba-like forcing constructed in Theorems 2 – 5?

Congratulations Peter and Philip! Thank you both so much for all you have done for me over the years!

Thank you all very much for your attention!