On hyper-stationary sets

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Colloquium and Workshop on the Occasion of the 60th Birthdays of Peter Koepke and Philip Welch
University of Bonn
23-25 May 2014

Topologies on ordinals

The difficulty to reach a limit ordinal number α may be measured by the richness of the coarsest topology (on ON or on some ordinal $\delta > \alpha$, extending the usual order-topology) for which α is an isolated point.

Topologies on ordinals

Recall that the order topology (also known as the interval topology) is the topology τ_0 generated by the set $\mathcal{B}_0$ consisting of $\{0\}$ and the intervals (α, β).

Notice that τ_0 is a Hausdorff scattered topology in which 0 and all successor ordinals are isolated points, and the accumulation points are the limit ordinals.

Thus, τ_0 is non-discrete on some ordinal δ if and only if $\delta > \omega$.

We shall define a sequence of topologies $\tau_0 \subseteq \tau_1 \subseteq \ldots \tau_\xi \subseteq \ldots$ (on some ordinal δ, or on ON), with τ_0 being the interval topology.

Topologies on ordinals

Given τ_ξ, let $d_\xi : \mathcal{P}(\delta) \to \mathcal{P}(\delta)$ be the Cantor derivative operator, defined by:

$$d_\xi(A) = \{\alpha < \delta : \alpha \text{ is a limit point of } A \text{ in the } \tau_\xi \text{ topology}\}.$$

Then let $\tau_{\xi+1}$ be the topology generated by

$$\mathcal{B}_{\xi+1} := \mathcal{B}_\xi \cup \{d_\xi(A) : A \subseteq \delta\}.$$

Notice that $d_0(A)$ is the set of limit points of A in the ordinal ordering. Thus, if the cofinality of α is uncountable and $\alpha \in d_0(A)$, then $d_0(A) \cap \alpha$ is a club (closed and unbounded) subset of α.

Topologies on ordinals

The set $\mathcal{B}_1 := \mathcal{B}_0 \cup \{d_0(A) : A \subseteq \delta\}$ is a base for the topology τ_1 on δ, known as the club topology.

Note that every $\alpha < \delta$ of countable cofinality is an isolated point of τ_1. Thus, τ_1 is non-discrete if and only if $\delta > \omega_1$.

What is $d_1(A)$?

Fact

For every $A \subseteq \delta$,

$$d_1(A) = \{\alpha : A \cap \alpha \textit{ is stationary in } \alpha\}.$$

Topologies on ordinals

Let us look now at the next topology, τ_2, which is generated by the set $\mathcal{B}_2 := \mathcal{B}_1 \cup \{d_1(A) : A \subseteq \delta\}$.

Notice that if some stationary subset S of α does not reflect (i.e., $d_1(S) = \{\alpha\}$), then α is an isolated point of τ_2. So, for τ_2 to be non-discrete topology we need at least that some $\alpha < \delta$ is stationary-reflecting, i.e., $d_1(S) \cap \alpha \neq \emptyset$, for all stationary $S \subseteq \alpha$.

It is well-known that the first stationary-reflecting cardinal, if it exists, must be either weakly inaccessible or the successor of a singular cardinal.

So if, e.g., $\delta \leq \aleph_{\omega+1}$, then τ_2 is discrete.

Topologies on ordinals

But for τ_2 to be non-discrete we need more than just the existence of a stationary-reflecting cardinal α. What we need is some α such that every pair A, B (and, in fact, every finite collection) of stationary subsets of α simultaneously reflect, that is, there exists $\beta < \alpha$ with $\beta \in d_1(A) \cap d_1(B)$.

Let us call such an α simultaneoulsy stationary-reflecting, or s-reflecting for short.

Topologies on ordinals

Proposition

$\mathcal{B}_2$ is a sub-base for a topology such that for every α, α is not isolated if and only if it is s-reflecting. Hence, τ_2 is a non-discrete topology on δ if and only if some $\alpha < \delta$ is s-reflecting.

Π^1_1-indescribable cardinals

Recall that a cardinal κ is Π^1_1-indescribable if for every $A \subseteq V_\kappa$ and every Π^1_1-sentence $\varphi(A)$, if

$$\langle V_\kappa, \in, A\rangle \models \varphi(A)$$

then there is $\lambda < \kappa$ such that

$$\langle V_\lambda, \in, A \cap V_\lambda\rangle \models \varphi(A \cap V_\lambda).$$

Theorem (Hanf and Scott 1961, and Keisler 1962)

A cardinal is Π^1_1-indescribable if and only if it is weakly-compact.

Topologies on ordinals

It is easy to see that every Π^1_1 indescribable is s-reflecting. Thus, in every model of set theory where there exists a weakly compact cardinal less than some limit ordinal δ, τ_2 is a non-discrete topology on δ.

Theorem (Jensen)

In the constructible universe L a regular cardinal κ is stationary-reflecting if and only if it is weakly compact, hence if and only if it is s-reflecting.[a]

[a] R. Jensen, The fine structure of the constructible hierarchy. Annals of Math. Logic 4 (1972)

Thus, in L, the topology τ_2 on some ordinal δ is non-discrete if and only if there exists a weakly-compact cardinal less than δ. The non-isolated points are precisely those ordinals whose cofinality is a weakly-compact cardinal.

ξ-stationary sets

In order to characterize the non-isolated points in the τ_ξ topology we need some definitions that generalize the notions of stationary set and stationary reflection.

Definition

Let δ be a limit ordinal. We say that $A \subseteq \delta$ is 0-stationary in α if and only if $A \cap \alpha$ is unbounded in α.
For $\xi > 0$, we say that A is ξ-stationary in $\alpha < \delta$ if and only if for every $\zeta < \xi$, every subset S of α that is ζ-stationary in α ζ-reflects to some $\beta \in A$, i.e., $S \cap \beta$ is ζ-stationary in β.

ξ-stationary sets

Note that A is 1-stationary in α if and only if $A \cap \alpha$ is stationary in α.

Fact

For every ξ,

$$d_\xi(A) = \{\alpha : A \cap \alpha \text{ is } \xi\text{-stationary in } \alpha\}.$$

ξ-stationary reflection

Definition

We say that a limit ordinal α is ξ-stationary-reflecting (ξ-reflecting, for short) if and only if $d_\zeta(S)$ is ζ-stationary in α, for every $\zeta < \xi$ and every $S \subseteq \alpha$ that is ζ-stationary in α. Equivalently, if and only if α is ξ-stationary in α.

Thus, α is 0-reflecting if and only if it is a limit ordinal; it is 1-reflecting if and only if it has uncountable cofinality; and it is 2-reflecting if and only if it is stationary-reflecting.

ξ-stationary reflection

Definition

We say that an ordinal α is ξ-simultaneously-stationary-reflecting (ξ-s-reflecting, for short) if and only for every $\zeta < \xi$, every pair of ζ-stationary subsets $A, B \subseteq \alpha$ simultaneously ζ-reflect at some $\beta < \alpha$, i.e., $A \cap \beta$ and $B \cap \beta$ are ζ-stationary in β.

Note that α is 0-s-reflecting if and only if it has uncountable cofinality; and it is 1-s-reflecting if and only if it is s-reflecting.

Fact

An ordinal α is ξ-s-reflecting if and only if $d_\zeta(A) \cap d_\zeta(B)$ is ζ-stationary in α, for every $\zeta < \xi$ and every ζ-stationary $A, B \subseteq \alpha$.

Characterizing non-isolated points

Theorem

For every ξ, an ordinal $\alpha < \delta$ is non-isolated in the τ_ξ topology on δ if and only if α is ξ-s-reflecting.

Question

Do ξ-s-reflecting ordinals exist?

If they do, how big are they?

Second-order indescribable cardinals

Recall that a cardinal κ is Π^1_n-indescribable if for every $A \subseteq V_\kappa$ and every Π^1_n-sentence $\varphi(A)$, if

$$\langle V_\kappa, \in, A\rangle \models \varphi(A)$$

then there is $\lambda < \kappa$ such that

$$\langle V_\lambda, \in, A \cap V_\lambda\rangle \models \varphi(A \cap V_\lambda).$$

Proposition

Every Π^1_n-indescribable cardinal is (n+1)-s-reflecting.

Indescribability and reflection

The point is that for every $n > 0$, the fact that $A \subseteq \kappa$ is n-stationary can be expressed as a Π^1_n sentence over $\langle V_\kappa, \in, A, \kappa\rangle$.

Thus, if the cofinality of $\alpha < \delta$ is a Π^1_n-indescribable cardinal, then α is a non-isolated point for the τ_{n+1} topology on δ.

Separating the topologies

Theorem

CON$(\exists\kappa<\lambda\,(\kappa$ *is* Π^1_n*-indescribable* $\wedge\ \lambda$ *is inaccessible*$))$ *implies CON*$(\tau_{n+1}$ *is non-discrete* $\wedge\ \tau_{n+2}$ *is discrete*$)$.

Proof.

Let κ be Π^1_n-indescribable, and let $\lambda>\kappa$ be inaccessible. In L, κ is n-indescribable and λ is inaccessible. So, in L, let κ_0 be the least $n+1$-s-reflecting cardinal, and let λ_0 be the least inaccessible cardinal above κ_0. Then L_{λ_0} is a model of ZFC in which κ_0 is $n+1$-s-reflecting and no ordinal greater than κ_0 is $n+2$-s-reflecting. In fact, no regular cardinal greater than κ_0 is 2-reflecting. The reason is that if α is a regular cardinal greater than κ_0, then $\alpha=\beta^+$, for some β. And since $\Box_\beta$ holds, there is a stationary subset of α that does not reflect. □

Reflection and indescibability in L

In L, a cardinal κ is Π^1_n-*indescribable* if for every $A \subseteq \kappa$ and every Π^1_n formula $\varphi(X)$, if

$$\langle L_\kappa, \in, A\rangle \models \varphi(A),$$

then there is $\lambda < \kappa$ such that

$$\langle L_\lambda, \in, A \cap \lambda\rangle \models \varphi(A \cap \lambda).$$

Jensen shows that in L a regular cardinal is 2-reflecting if and only if it is Π^1_1-indescribable, and therefore if and only if it is 2-s-reflecting.
We have the analogous result for $(n+1)$-reflecting cardinals and Π^1_n-indescribability, for all $n > 0$.

Reflection and indescibability in L

Theorem (J.B., M. Magidor, and H. Sakai)

Assume $V = L$. For every $n > 0$, a regular cardinal is $n+1$-stationary if and only if it is $n+1$-reflecting, if and only if it is $n+1$-s-reflecting, if and only if it is Π^1_n-indescribable.

Thus, in L the non-isolated points of the τ_{n+1} topology are precisely the ordinals whose cofinality is a Π^1_n-indescribable cardinal.

The proof actually shows that for every $(n+1)$-stationary $S \subseteq \kappa$, every Π^1_n sentence true in L_κ reflects to some ordinal in S. The case $n = 1$ is due to Jensen.

The ideal of non-ξ-stationary sets

For each limit ordinal α and each ξ, let $\mathcal{I}^\xi_\alpha$ be the set of non-ξ-stationary subsets of α, and let

$$\mathcal{F}^\xi_\alpha = (\mathcal{I}^\xi_\alpha)^* := \{A \subseteq \alpha : \alpha - A \in \mathcal{I}^\xi_\alpha\}.$$

Thus, if α has uncountable cofinality, then $\mathcal{I}^1_\alpha$ is the ideal of non-stationary subsets of α and $\mathcal{F}^1_\alpha$ is the club filter over α.

Proposition

For every ξ, an ordinal α is ξ-s-reflecting if and only if $\mathcal{I}^\xi_\alpha$ is a proper ideal, hence if and only if $\mathcal{F}^\xi_\alpha$ is a proper filter.

Proposition

If κ is a Π^1_n-indescribable cardinal, then the filter $\mathcal{F}^{n+1}_\kappa$ is normal and κ-complete.

On the consistency strength of ξ-stationarity

Definition (Mekler-Shelah, 1989)

A regular uncountable cardinal κ is a reflection cardinal if there exists a proper normal ideal $\mathcal{I}$ on κ such that for every $X \subseteq \kappa$,

$$X \in \mathcal{I}^+ \quad \Rightarrow \quad d_1(X) \in \mathcal{I}^+.$$

Note that if κ is 2-reflecting, then NS_κ is the smallest such ideal.

Theorem (Mekler-Shelah, 1989)

The following are equiconsistent:

1. *There exists κ 2-reflecting.*
2. *There exists a reflection cardinal.*

On the consistency strength of ξ-stationarity

They show that if κ is a reflection cardinal, then in some generic extension of *L*, κ is 2-reflecting (in fact, *Reg* is 2-stationary on κ).

Moreover, they show that in *L*, if κ is at most the first greatly-Mahlo cardinal, then κ is not a reflection cardinal. Thus, the first reflection cardinal is strictly between the first greatly-Mahlo cardinal and the first weakly-compact.

Conjecture

The following are equiconsistent for n > 1:

1. *There exists κ n-reflecting.*
2. *There exists an n-reflection cardinal.*

Happy Birthday!