A Simple $P_{\aleph_1}$-Point and a Simple $P_{\aleph_2}$-Point

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Filters

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We call a filter non-principal if it contains all cofinite sets.

We call it an ultrafilter if it is a maximal filter.

P_κ-points, filter bases, and simple filters

Definition

Let κ be a regular uncountable cardinal.

(1) An ultrafilter $\mathscr{W}$ is called a P_κ-point if for every $\gamma < \kappa$, for every $A_i \in \mathscr{U}$, $i < \gamma$, there is some $A \in \mathscr{W}$ such that

$$\forall i < \gamma A \subseteq^* A_i.$$

Such an A is called a pseudo-intersection of the A_i, $i < \gamma$.

A $P_{\aleph_1}$-point is called a P-point.

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(2) A subset of a filter $\mathscr{W}$ is called a basis of $\mathscr{W}$ if every set in $\mathscr{W}$ has a subset that is in the basis.

(3) An ultrafilter is called a simple P_κ-point if it has a basis of the form $\{B_\alpha \mid \alpha < \kappa\}$ such that $B_\beta \subseteq^* B_\alpha$ for $\alpha < \beta < \kappa$.

An old theorem

Theorem

It is consistent relative to ZFC *that there is a simple* $P_{\aleph_1}$*-point and a simple* $P_{\aleph_2}$*-point.*

In such a forcing extension we have $\mathfrak{u} < \mathfrak{s}$ and exactly two near coherence classes of ultrafilters.

Preserving ultrafilters

Let $\mathbb{P}$ be a notion of forcing. We say that $\mathbb{P}$ preserves an ultrafilter $\mathscr{W}$ over I if

$$\Vdash_{\mathbb{P}} \text{“}(\forall X \subseteq I)(\exists Y \in \mathscr{W})(Y \subseteq X \vee Y \subseteq I \setminus X)\text{”}$$

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In the first case

$$\{X \in [\omega]^\omega \cap \mathbf{V}[G] \mid (\exists Y \in \mathscr{W}) X \supseteq Y\}$$

is an ultrafilter in $\mathbf{V}[G]$.

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Covering ω-sequences:
If $\mathbb{P}$ is proper and preserves $\mathscr{W}$ and $\mathscr{W}$ is a P-point, then $\mathscr{W}$ stays a P-point.

The Rudin-Blass ordering

Definition

Let $\mathscr{F}$, $\mathscr{F}'$ be non-principal filters over ω. We write $\mathscr{F} \leq_{\mathrm{RB}} \mathscr{F}'$ and say $\mathscr{F}$ is Rudin-Blass-below $\mathscr{F}'$ iff there is a finite-to-one f such that $f(\mathscr{F}) = \{X \mid f^{-1}[X] \in \mathscr{F}\} \subseteq f(\mathscr{F}')$.

From blocksequences down to their content

Definition

$\mathrm{set}(\bar{a}) = \bigcup_{n<\omega} a_n$.

$\Phi(\mathscr{F}) = \{\mathrm{set}(\bar{a}) \mid \bar{a} \in \mathscr{F}\}$

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Wish:
A simple $P_{\aleph_2}$-point will be growing in an increasing chain of non-rapid P-points $\mathscr{R}_\alpha = \Phi(\mathscr{F}_\alpha)$ over ω, $\alpha \leq \omega_2$.

Let the $\mathscr{F}_\alpha$, $\alpha \leq \omega_2$, be $\subseteq$-increasing.

How do we arrange the $\mathscr{F}_\alpha$?

Find $\mathscr{F}_\alpha \in \mathbf{V}^{\mathbb{P}_\alpha}$.

Approximations to the $P_{\aleph_2}$-point plus additional structure

Find $\mathscr{F}_\alpha \in \mathbf{V}^{\mathbb{P}_\alpha}$.

We let $\mathbb{Q}_\alpha = \mathbb{BS}(\mathscr{F}_\alpha)$,
a forgetful σ-centred version of Blass–Shelah forcing in which the second components are elements of $\mathscr{F}_\alpha$.

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The generic real μ_α for $\mathbb{Q}_\alpha$, will diagonalise $\Phi(\mathscr{F}_\alpha) = \mathscr{R}_\alpha$ and be an element of $\Phi(\mathscr{F}_{\alpha+1})$.

Block-sequences with norms

Definition

(1) We let $\mathbb{F}_{\text{lm}} = \{a \subseteq \omega \mid a \text{ finite, non-empty}\}$ denote the set of blocks.

(2) For $a \in \mathbb{F}_{\text{lm}}$ we let $||a|| = \min(\lg(|a|), \min(a))$.

(3) For $a, b \in \mathbb{F}_{\text{lm}}$ we write $a < b$ if $(\forall n \in a)(\forall m \in b)(n < m)$.

(4) A sequence $\bar{a} = \langle a_n \mid n \in \omega \rangle$ of members of $\mathbb{F}_{\text{lm}}$ is called unmeshed if for all n, $a_n < a_{n+1}$.

(5) $\bar{a} = \langle a_n \mid n \in \omega \rangle$ of members of $\mathbb{F}_{\text{lm}}$ is called diverging if

$$\lim_{n \to \omega} ||a_n|| = \infty.$$

(6) By $(\mathbb{F}_{\text{lm}})^{\omega}$ we denote the set of unmeshed diverging sequences of members in $\mathbb{F}_{\text{lm}}$.

Dropping and merging

Definition

(7) If X is a subset of $\mathbb{F}_{\mathrm{lm}}$, we write $\mathrm{FU}(X)$ for the set of all finite unions of members of X. Let $\bar{a} = \langle a_n \mid n < \omega \rangle$. We write $\mathrm{FU}(\bar{a})$ instead of $\mathrm{FU}(\{a_n \mid n \in \omega\})$. We write $b \in \bar{a}$ for $b \in \{a_n \mid n \in \omega\}$.

(8) For $\bar{a} \subseteq \mathbb{F}_{\mathrm{lm}}$, the set $(\mathrm{FU}(\bar{a}))^\omega$ denotes the collection of all infinite unmeshed diverging sequences in $\mathrm{FU}(\bar{a})$.

Dropping and merging allows to define Matet forcing.

We now add a third operation: Taking sub-blocks, and even splitting blocks.

The partial orders of condensation and strengthening

Definition

(1) Given $\bar{a}, \bar{b} \in (\mathbb{F}_{\rm lm})^\omega$, we say that $\bar{b}$ is a condensation of $\bar{a}$ and we write $\bar{b} \sqsubseteq \bar{a}$ if $\bar{b} \in (\mathrm{FU}(\bar{a}))^\omega$. In other words, each b_n is a union of finitely many, at least one $a_{i_1}, \dots, a_{i_k}$. We say $\bar{b}$ is almost a condensation of $\bar{a}$ and we write $\bar{b} \sqsubseteq^* \bar{a}$ iff there is a k such that $\langle b_n \mid k \leq n < \omega \rangle$ is a condensation of $\bar{a}$.

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(2) Given $\bar{a}$ and $\bar{b} \in (\mathbb{F}_{\mathrm{lm}})^\omega$, we say that $\bar{b}$ is a strengthening of $\bar{a}$ and we write $\bar{b} \leq \bar{a}$ if each b_n is a non-empty subset of a union of finitely many a_i. We can equivalently say $\mathrm{set}(\bar{b}) \subseteq \mathrm{set}(\bar{a})$. We say $\bar{b}$ is almost a strengthening of $\bar{a}$ and we write $\bar{b} \leq^* \bar{a}$ iff there is k such that $\langle b_n \mid k \leq n < \omega \rangle$ is a strengthening of $\bar{a}$.

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(3) Given X and $Y \in (\mathbb{F}_{\mathrm{lm}})^\omega$, we write $X =^* Y$ if they coincide up to finitely many blocks.

Useful notions of filters

Definition

(1) A filter over $\mathbb{F}_{\mathrm{lm}}$ is a non-empty subset of $\mathcal{P}(\mathbb{F}_{\mathrm{lm}})$ that is $\subseteq$-upwards closed and closed under finite intersections and does not contain the empty set.

(2) A filter (in $(\mathbb{F}_{\mathrm{lm}})^\omega$) is a non-empty subset of $(\mathbb{F}_{\mathrm{lm}})^\omega$ that is closed under $\leq$-larger elements and under $=^*$ such that for any two elements in the filter there is a common $\leq$-lower bound in the filter.

Why do we allow to break up blocks?

Lemma

If $\bar{a} \in (\mathbb{F}_{\mathrm{lm}})^\omega$ *and* $X \subseteq \omega$ *then there is* $\bar{b} \leq \bar{a}$*,* $\bar{b} \in (\mathbb{F}_{\mathrm{lm}})^\omega$ *such that* $\mathrm{set}(\bar{b}) \subseteq X$ *or* $\mathrm{set}(\bar{b}) \subseteq (\omega \setminus X)$*.*

We need to know more about filter for Ramsey theoretic computations

Definition

(1) Let $A \subseteq \mathbb{F}_{\mathrm{lm}}$, $b \in \mathbb{F}_{\mathrm{lm}}$. We write $(A \text{ past } b)$ for $\{c \in A \mid \min(\mathrm{dom}(c)) \geq \max(\mathrm{dom}(b))\}$. and also for the increasing enumeration of the latter.

Now let $\mathscr{F}$ be a filter in $(\mathbb{F}_{\mathrm{lm}})^\omega$.

(2) $\mathscr{F}$ contains diagonal lower bounds if for any $\leq^*$-descending sequence $\langle \bar{a}_n \mid n \in \omega \rangle$ of members of $\mathscr{F}$ there is a $\bar{b} \in \mathscr{F}$ such that

$$(\forall c \in \bar{b})((\bar{b} \text{ past } c) \leq \bar{a}_{\max(c)+1}).$$

Such a $\bar{b}$ is called a diagonal lower bound of $\langle \bar{a}_n \mid n \in \omega \rangle$.

(3) $\mathscr{F}$ is said to have the Hindman property iff: For every $\bar{a} \in \mathscr{F}$, $k \geq 1$ and $c \colon \mathrm{FU}(\bar{a}) \to k$ there is $\bar{b} \leq \bar{a}$, $\bar{b} \in \mathscr{F}$, such that $c \restriction \mathrm{FU}(\bar{b})$ is monochromatic.

Continuation of desirable properties

Definition

(4) A filter $\mathscr{F} \subseteq (\mathbb{F}_{\mathrm{lm}})^{\omega}$ is called full iff $\Phi(\mathscr{F}) = \{\mathrm{set}(\bar{a}) \mid \bar{a} \in \mathscr{F}\}$ is an ultrafilter over ω.

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(5) A filter $\mathscr{F} \subseteq (\mathbb{F}_{\mathrm{lm}})^\omega$ is called maximal iff for any $\bar{a} \in (\mathbb{F}_{\mathrm{lm}})^\omega \setminus \mathscr{F}$ there is $\bar{b} \in \mathscr{F}$ that is $\leq$-incompatible with $\bar{a}$.

Very suitable filters

From now on we fix a P-point $\mathscr{E}$ in the ground model, and we assume that CH holds.

Definition

(1) A suitable filter is a maximal full filter $\mathscr{F} \subseteq (\mathbb{F}_{\mathrm{lm}})^{\omega}$ that contains diagonal lower bounds and has the Hindman property.

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(2) A very suitable filter is a maximal full filter $\mathscr{F} \subseteq (\mathbb{F}_{\mathrm{lm}})^\omega$ that contains diagonal lower bounds, has the Hindman property and such that $\Phi(\mathscr{F}) \not\leq_{\mathrm{RB}} \mathscr{E}$.

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Lemma

Under CH, *given a P-point $\mathscr{E}$, there is a very suitable filter.*

Hindman's theorem

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Let $\bar{a} \in (\mathbb{F}_{\mathrm{lm}})^{\omega}$. Let $k \geq 1$ be a natural number and let $c\colon \mathrm{FU}(\bar{a}) \to k$. Then there is $\bar{b} \sqsubseteq \bar{a}$, $\bar{b} \in (\mathbb{F}_{\mathrm{lm}})^{\omega}$, such that $c \restriction \mathrm{FU}(\bar{b})$ is monochromatic.

Very suitable selective coideals

Definition

A set $\mathscr{H} \subseteq (\mathbb{F}_{\mathrm{lm}})^\omega$ is called a very suitable selective coideal in $((\mathbb{F}_{\mathrm{lm}})^\omega, \sqsubseteq, \leq)$ if the following hold:

(1) (Upwards Closure)
$\mathscr{H} \subseteq (\mathbb{F}_{\mathrm{lm}})^\omega$, and $\bar{a} \in \mathscr{H}$ and $\bar{b} \geq \bar{a}$ implies $\bar{b} \in \mathscr{H}$,

(2) (Diagonal Lower Bounds)
If $\langle A_i \mid i \in \omega \rangle$ is a $\leq^*$-descending sequence of elements $A_i \in \mathscr{H}$, then there is $B \in \mathscr{H}$ such that $(\forall b \in B) B \text{ past } (b)) \leq A_{\max(b)+1}$.

(3) (Hindman)
For every $\bar{a} \in \mathscr{H}$, $k \geq 1$ and $c\colon \mathrm{FU}(\bar{a}) \cap \mathscr{H} \to k$ there is $\bar{b} \leq \bar{a}$, $\bar{b} \in \mathscr{H}$, such that $c \restriction \mathrm{FU}(\bar{b})$ is monochromatic.

Very suitable selective coideals continued

Definition

(4) (Fullness)
If $X \subseteq \omega$, $\bar{a} \in \mathscr{H}$, then $\exists \bar{b} \in \mathscr{H}$, $\bar{b} \leq \bar{a}$ such that $\mathrm{set}(\bar{b}) \subseteq X$ or $\mathrm{set}(\bar{b}) \subseteq (\omega \setminus X)$.

(5) (Genericity)
If $\bar{a} \in (\mathbb{F}_{\mathrm{lm}})^\omega$, and D is open dense in the $\leq$-preorder in $(\mathbb{F}_{\mathrm{lm}})^\omega$, then $\exists \bar{b} \in \mathscr{H} \cap D$ with $\bar{b} \leq \bar{a}$.

(6) ($\not\leq_{RB} \mathscr{E}$)
If h is a finite-to-one function then there is $E \in \mathscr{E}$ and $\bar{a} \in \mathscr{H}$ such that $h[\mathrm{set}(\bar{a})] \cap h[E] = \emptyset$.

A proposition about happy families

Definition

A filter $\mathscr{F}$ is generated by $\langle \bar{a}^\xi \mid \xi < \zeta \rangle$ iff
$\mathscr{F} = \{\bar{b} \in (\mathscr{F}_{\mathrm{lm}})^\omega \mid \exists \xi < \zeta \bar{a}^\xi \leq^* \bar{b}\}$.

Prop. 0.11, Mathias, Happy Families

Assume CH. If $\mathscr{H}$ is a very suitable selective coideal in $(\mathbb{F}_{\mathrm{lm}})^\omega$ then there is a very suitable filter $\mathscr{F} \subseteq \mathscr{H}$.

More Ramsey theory: Taylor's theorem

Taylor

Let $\mathscr{F}$ be a suitable filter, $n \in \omega$. Let $[\mathrm{FU}(\bar{a})]^n_<$ be partitioned into finitely many sets. Then there is $\bar{b} \in \mathscr{F}$, $\bar{b} \leq \bar{a}$, such that $[\mathrm{FU}(\bar{b})]^n_< \cap [\mathscr{F}]^n$ lies in that part of the partition.

Blass–Shelah forcing $\mathbb{BS}$

(A forgetful variant.)

Definition

In the Blass–Shelah forcing with logarithmically measured blocks, $\mathbb{BS}$, the conditions are pairs $(s, \bar{a})$ such that $s \in \mathscr{P}_{<\omega}(\omega)$ and $\bar{a} \in ((\mathbb{F})_{\mathrm{lm}})^{\omega}$ and $s < a_0$.

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The forcing order is $(t, \bar{b}) \leq (s, \bar{a})$ (recall the stronger condition is the smaller one) iff

(1) $s \subseteq t$ and

(2) $t \smallsetminus s$ is a subset of a union of finitely many a_n and

(3) $\bar{b}$ is a strengthening of $\bar{a}$.

A σ-centred subforcing, $\mathbb{BS}(\mathscr{F})$

Definition

Given a suitable filter $\mathscr{F}$ in $(\mathbb{F}_{\mathrm{lm}})^\omega$, Blass–Shelah forcing relative to $\mathscr{F}$, $\mathbb{BS}(\mathscr{F})$ consists of all pairs $(s, \bar{a})$ such that $\bar{a} \in \mathscr{F}$ and $s < \min(a_0)$.

The forcing order is as follows: is $(t, \bar{b}) \leq (s, \bar{a})$ iff

(1) $s \subseteq t$ and

(2) $t \setminus s$ a union of finitely many a_n and

(3) $\bar{b}$ is a strengthening of $\bar{a}$.

Note the difference between the two definitions: In $\mathbb{BS}(\mathscr{F})$ we work with FU for the choice of $t \setminus s$, and splitting blocks occurs just when strengthening the second component within $\mathscr{F}$.

The effect of $\mathbb{BS}(\mathscr{F})$

Definition

Let G be $\mathbb{BS}(\mathscr{F})$-generic over $\mathbf{V}$. We call

$$\mu = \bigcup \{w \mid \exists \bar{a}(w, a) \in G\}$$

the $\mathbb{BS}(\mathscr{F})$-generic real.

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Remark

(1) Let $\mathscr{F}$ be a suitable filter. Let G be $\mathbb{BS}(\mathscr{F})$-generic over $\mathbf{V}$. Let μ be defined from G as above. Then $G = \{(w, \bar{a}) \mid \mu \subseteq \mathrm{set}(\bar{a}) \cup w\}$.

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(2) $\mathbb{BS}(\mathscr{F})$ destroys the $\leq^*$-unboundedness of the family of enumerating functions of members of $\Phi(\mathscr{F})$. It preserves the $\leq^*$-unboundedness of the family of enumerating functions of $\mathscr{E}$.

Not too slim subsets of the generic real are generic as well

Remark

There is an analogue to Mathias' result about Mathias forcing with selective ultrafilters over ω*: Let* μ *be a generic real and let* $\mu' \subseteq \mu$ *be such that* $\forall \bar{a} \in \mathscr{F}\ \bar{a} \restriction \mu' \in (\mathbb{F}_{\mathrm{lm}})^\omega$*. Then* μ' *is* $\mathbb{BS}(\mathscr{F})$*-generic as well.*

Adding an unsplit real

Lemma

If $X \subseteq \omega$, $X \in \mathbf{V}$ *then after forcing with* $\mathbb{BS}(\mathscr{F})$ *we have* $\mu \subseteq^* X$ *or* $\mu \subseteq^* (\omega \setminus X)$.

What $\mathbb{BS}(\mathscr{F})$ does not do

Eisworth for Matet forcing relative to a Milliken Taylor ultrafilter $\mathscr{U}$.

Theorem

Let $\mathscr{F}$ be an very suitable filter, so $\Phi(\mathscr{F}) \not\leq_{\mathrm{RB}} \mathscr{E}$. Then we have: $\mathscr{E}$ continues to generate an ultrafilter after we force with $\mathbb{BS}(\mathscr{F})$.

Outline of an iteration

We construct by induction on $\alpha \leq \omega_2$ a countable support iteration of proper forcings $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \mid \beta < \omega_2, \alpha \leq \omega_2 \rangle$,
and simultaneously sequences of $\mathbb{P}_\beta$-names $\underset{\sim}{\mathscr{F}}_\beta$ for suitable filters (note that the space $((\mathbb{F}_{\mathrm{lm}})^\omega, \sqsubseteq, \leq)$ depends on the stage)
and relations $\bar{R}_\beta$ $\beta \leq \omega_2$ such that for any $\alpha \leq \omega_2$, the initial segment $\langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}_\beta, \mathscr{F}_\gamma, \bar{R}_\gamma, \bar{g}_{\gamma,a} \mid \beta < \alpha, \gamma \leq \alpha \rangle$ fulfils:

Continuation

(P1) For all $\gamma < \alpha$,

$$\Vdash_{\mathbb{P}_\gamma} \text{“}\underset{\sim}{\mathbb{Q}}_\gamma = \mathbb{BS}(\mathscr{F}_\gamma) \text{ for a suitable filter } \mathscr{F}_\gamma \text{ adding } \mu_\gamma \ \wedge \ \Phi(\mathscr{F}_\gamma) \not\leq_{\mathrm{RB}} \mathscr{E}\text{”}$$

(P2) $\mathbb{P}_\alpha$ is proper and

$$\mathbb{P}_\alpha \Vdash \text{“}\mathscr{E} \text{ generates an ultrafilter”.}$$

(P3) $\mathbb{P}_\alpha \Vdash (\forall \gamma < \alpha)\mathscr{F}_\gamma \subseteq \mathscr{F}_\alpha$ If $\mathrm{cf}(\alpha) \leq \omega$, then we require in addition $\mathbb{P}_\alpha \Vdash \exists \bar{b} \in \mathscr{F}_\alpha \, \mathrm{set}(\bar{b}) = \delta_\alpha$.

(P4) $(\bar{R}, \mathscr{S}, \bar{g})$ covers at stage α and $\mathbb{P}_\alpha$ is $(\bar{R}, \mathscr{S}, \bar{g})$-preserving.

Complementing filters to suitable filters

The talk ended before this slide, because the time of 50 minutes was over.

Definition

(a) $\mathbf{V}_\alpha = \mathbf{V}^{\mathbb{P}_\alpha}$ stands for an arbitrary $\mathbb{P}_\alpha$-generic extension of $\mathbf{V} = \mathbf{V}_0$.

(b) Let $\alpha \leq \omega_2$, $\mathscr{B} \in \mathbf{V}_\alpha$ be set such that any finitely many elements are $\leq$-compatible in $(\mathbb{F}_{\mathrm{lm}})^\omega$ (in $\mathbf{V}_\alpha$ or in $\mathbf{V}_\beta$, this is equivalent). We let

$$(\mathscr{B})^{+\alpha} = \{\bar{a} \in (\mathbb{F}_{\mathrm{lm}})^\omega \cap \mathbf{V}_\alpha \mid \forall \bar{b} \in \mathscr{B} \exists \bar{c} \in (\mathbb{F}_{\mathrm{lm}})^\omega \;\; \bar{c} \leq \bar{a}, \bar{b}\},$$

be the coideal of sequences in $\mathbf{V}_\alpha$ that are compatible with $\mathscr{B}$ in $\mathbf{V}_\alpha$.

Towards a simple ultrafilter: a $\subseteq^*$-descending sequence

In the hard case $\mathrm{cf}(\alpha) = \omega$ we first need a diagonal intersection of cofinally many former generic reals:

Definition

Let $\langle \gamma_n \mid n < \omega \rangle$ be increasing and cofinal in α. Let δ_α be a diagonal intersection of μ_{γ_n}, $n \in \omega$.

We will prove that there is a very suitable filter

$$\mathscr{F}_\alpha \supseteq \{\bar{b} \restriction \delta_\alpha \mid \bar{b} \in \bigcup_{\gamma < \alpha} \mathscr{F}_\gamma\}.$$

Extending a filter to a maximal filter

Lemma

Let $\mathscr{R}' \subseteq (\mathbb{F}_{\mathrm{lm}})^\omega$ *be a filter and let* $\mathscr{R}$ *be a maximal filter.*
$\mathscr{R} \subseteq (\mathscr{R}')^+$ *iff* $\mathscr{R} \supseteq \mathscr{R}'$*.*

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We apply this lemma in $\mathbf{V}_\alpha$ with

$$\begin{aligned} \mathscr{R}' &= \{\bar{b} \restriction \delta_\alpha \mid \bar{b} \in \bigcup_{\gamma<\alpha} \mathscr{F}_\gamma\} \text{ and} \\ \mathscr{R} &= \mathscr{F}_\alpha. \end{aligned}$$

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Big wish: $(\mathscr{R}')^+$ is a very suitable coideal in $\mathbf{V}_\alpha$.

The relations we want to preserve

Definition

Assume that $\langle \mathscr{F}_\gamma \mid \gamma < \alpha \rangle$ is an increasing sequence of very suitable filters $\mathscr{F}_\gamma \in \mathbf{V}_\gamma$ and in $\mathbf{V}_\gamma$. We say $f R_{n,\alpha} \bar{g}$ if the following holds in $\mathbf{V}_\alpha$:

(1) $f = (\bar{A}, h, c, X)$, for $n \in \omega$,

(2) $\bar{A} = \langle A_\ell \mid \ell \in \omega \rangle$ is a $\leq$-descending sequence of elements $A_\ell = \bar{a}^\ell = \langle a_n^\ell \mid n \in \omega \rangle$ of $(\mathbb{F}_{\mathrm{lm}})^\omega$ such that

$$\mathrm{set}(A_\ell) \subseteq \delta_\alpha$$

$$(\forall Y \in \bigcup_{\gamma<\alpha} \mathscr{F}_\gamma))$$

there is a common strengthening Y' of A_ℓ and of Y that is a member of $(\mathbb{F}_{\mathrm{lm}})^\omega$),

Continuation 1: Def. of $f R_{n,\alpha} \bar{g}$

Definition

(3) h is finite-to-one,

(4) $c\colon \mathbb{F}_{\mathrm{lm}} \to k$ for some finite k,

(5) $X \subseteq \omega$,

(6) $\bar{g} \in (\mathbb{F}_{\mathrm{lm}})^{\omega}$,

Definition

(7)

$$(\forall \ell \in \omega)\Big(((\bar{g} \text{ past } g_\ell),\ \text{past } n+1) \leq A_{\max(g_\ell)+1}\Big) \wedge$$
$$(\mathrm{set}(\bar{g}) \cap [n,\infty) \subseteq X \vee \mathrm{set}(g) \cap [n,\infty) \subseteq \omega \setminus X \wedge \mathrm{set}(\bar{g}) \subseteq \delta_\alpha) \wedge$$
$c \restriction \mathrm{FU}(g \restriction [n,\infty))$ is monochromatic $\wedge$

$(\forall Y \in \bigcup_{\gamma<\alpha} \mathscr{F}_\gamma)$ $\bar{g}$ and Y are compatible in $(\mathbb{F}_{\mathrm{lm}})^\omega \wedge$

$$\exists E \in \mathscr{E} h[\mathrm{set}(g)] \cap h[E] \subseteq n.$$

Benefitting from properness

Definition

Under CH, we let $N \prec H(\chi)$, $N \cap \omega_1 = a$. (Under CH, $N \cap H(\omega_1)$ is determined by $N \cap \omega_1$, and our definition makes sense only under this condition.)

We say $f R_{n,\alpha,a} \bar{g}$ if the modification of $f R_{n,\alpha} \bar{g}$ by writing the quantifiers $(\forall Y \in N \cap \bigcup_{\gamma<\alpha}(\{\bar{a} \restriction \mu_\gamma \mid \bar{a} \in \mathscr{F}_\gamma\}))$ and $\exists E \in \mathscr{E} \cap N$ in items (2) and (7) holds.

There are coverers

Lemma

Suppose that $\mathbb{P}_\gamma$ *is* $(\bar{R}, \mathscr{S}, \bar{g})$*-preserving for stage* γ *for* $\gamma < \alpha$*, and* $\mathscr{F}_\gamma$ *are suitable filters, increasing with* γ*. Then there for every* $N \in \bigcup_{\gamma<\alpha} \mathbf{V}_\gamma$ *with* $\mathbb{P}_\alpha \in N$ *there is* $\bar{g}_{\alpha,a}$ *such that for any* $f \in N$ $\exists n \; f R_{n,\alpha,a} \bar{g}_{\alpha,a}$.

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Now: If $N \in \bigcup_{\gamma\leq\alpha} \mathbf{V}_\gamma$?
Does $\bar{g}$ still cover?

A preservation theorem

Theorem

Suppose that $(\bar{R}, \mathscr{S}, \bar{g})$ *strongly covers in* $\mathbf{V}$. *Suppose* CH *and* $\alpha \leq \aleph_2$. *Let* $\alpha \leq \omega_2$ *and let* $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \mathbb{BS}(\mathscr{F}_\gamma) \mid \gamma < \alpha, \beta \leq \alpha \rangle$ *be the countable support iteration of proper iterands. Let* $\mathscr{F}_\gamma$ *be a very suitable filter in* $(\mathbb{F}_{\mathrm{lm}})^\omega$ *in* $\mathbf{V}_\gamma$, *increasing in* γ. *Let* $\mathscr{E}$ *be a* P*-point in* $\mathbf{V}_{\alpha_0}$, *not nearly coherent to* $\Phi(\mathscr{F}_\gamma)$ *in* $\mathbf{V}_\gamma$. *Then the following hold for* $\alpha \leq \omega_2$,

(1) $(\bar{R}, \mathscr{S}, \bar{g})$ *covers at stage* α *and* $\mathbb{P}_\alpha$ *is* $(\mathscr{S}, \bar{R}, \bar{g})$*-preserving,*

(2) for $\mathrm{cf}(\alpha) \leq \omega$, *in* $\mathbf{V}_\alpha$, $\{\bar{b} \restriction \delta_\alpha \mid \bar{b} \in \bigcup_{\gamma \in \alpha} \mathscr{F}_\gamma\}^{+\alpha}$ *is a very suitable selective coideal,*

(3) in $\mathbf{V}_\alpha$, *there is an very suitable filter* $\mathscr{F}_\alpha \supseteq \bigcup_{\gamma < \alpha} \mathscr{F}_\gamma$ *such that for* $\mathrm{cf}(\alpha) \leq \omega$

$$\exists \bar{b} \in \mathscr{F}_\alpha \, \mathrm{set}(\bar{b}) \subseteq \delta_\alpha.$$

Diagonal lower bounds in $(\mathbb{F}_{\mathrm{lm}})^\omega, \leq, \mathrm{past}\ )$ are not diagonal intersections in $\mathcal{P}(\omega)$

Remark

$\mathscr{R}$ is not rapid.

Proof: None of the $\mathscr{R}_\alpha = \Phi(\mathscr{F}_\alpha)$ for $\alpha < \omega_2$ is a rapid, since diagonalising a rapid ultrafilter means adding a dominating real and hence destroying any ultrafilter in the ground model, in particular $\mathscr{E}$. If $\mathscr{R}$ were rapid, this property would reflect to an ω_1-club of $\alpha < \omega_2$.